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Research paper

Analytical description of the structure of special discontinuities described by a generalized KdV–Burgers equation

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ABSTRACT

Traveling wave solutions of a generalized KdV–Burgers equation are studied. The nonlinearity is specified as a piecewise linear flux function consisting of four parts. A boundary value shock-structure problem is solved. An analytical solution describing the structures of special discontinuities (undercompressive shocks) is obtained, and the behavior of these solutions is examined for various parameters of the problem.

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1. Introduction

Discontinuous solutions of the generalized Hopf equation

$$\frac{\partial v}{\partial t}+\frac{\partial \varphi(v)}{\partial x}=0 \tag{1}$$

were studied in [1–5] in the case where the flux function $\varphi(v)$ has a single inflection point and in [6–12] in the case where $\varphi(v)$ has two points of inflection.

It is well known (see, e.g., [13]) that the Riemann problem for Eq. (1) can have an infinite set of solutions. The Riemann problem is an initial value one with initial conditions specified by a piecewise constant function having only one discontinuity. In many cases, the problem has a unique solution if the latter is constructed using only discontinuities with a structure (see, e.g., [12,14]).

To study discontinuity structures, we use a generalized (in the sense of nonlinearity) KdV–Burgers (KdVB) equation, which involves terms describing small-scale processes, such as dispersion and dissipation:

$$\frac{\partial v}{\partial t}+\frac{\partial \varphi(v)}{\partial x}=\mu\frac{\partial^2 v}{\partial x^2}-m\frac{\partial^3 v}{\partial x^3}, \tag{2}$$

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$$m, \mu = \text{const}, \qquad m > 0, \qquad \mu > 0, \qquad v = v(x, t).$$

On the right-hand side of (2), the term involving the coefficient m describes the dispersion effects (m is the dispersion parameter). The term involving the coefficient μ takes into account the viscous effects and dissipation (μ is the dissipation parameter). The nonlinearity is specified by the flux function $\varphi(v)$.

For Eqs. (1) and (2), the set of admissible values v_- behind a discontinuity that correspond to a given value v_+ ahead of the discontinuity was considered in [6]. It was shown that if $\varphi(v)$ is not convex and has two inflection points, then the set of admissible v_- consists of segments and isolated points. The number of isolated points and segments depends on μ/m^2, specifically, it increases indefinitely with growing μ/m^2. The solution of the Riemann problem becomes nonunique for some pairs of v, and the maximum number of solutions turns out to be equal to the number of isolated points. These results were described in more detail in [15,16].

It was shown in [6] that the isolated points from the set of admissible v_- behind a discontinuity with a structure are associated with certain values of the discontinuity velocity, which represents an additional boundary condition required for the discontinuity to be of evolution type [17]. Discontinuities corresponding to isolated points were referred to as *special discontinuities*.

A special discontinuity is one that satisfies relations following from conservation laws and obeys additional relations required for the existence of its structure. On the phase plane, the structure of such discontinuities for Eq. (2) is represented by an integral curve joining two saddles.

Special discontinuities, for example, combustion (deflagration) waves, have long been known. Structures of special discontinuities that solve Eq. (2) are also known as *undercompressive shock waves* [3] or *nonclassical shock waves* [2]. We will use the term *special discontinuities*, which was introduced in [6]. Note that, in the general case, there are special discontinuities that are not shock waves. Additionally, in [1–5,13] the solution of Eq. (2) corresponding to a stationary discontinuity structure was referred to as a *travelling wave solution*. Along with this terminology, following Godunov [18], we will use the term *structure of discontinuity*, or shock structure.

For Eq. (2) with $\varphi(v) = v^3$, a solution in the form of a special discontinuity was first obtained in an analytical form in [3]. For $\varphi(v) = v^2/(v^2 + a(1-v)^2)$, the structure of a special discontinuity was numerically found and studied in [2]. Both these flux functions $\varphi(v)$ have a single inflection point, and only one special discontinuity with a structure can exist for given parameters m and μ. In [5] a piecewise linear flux function consisting of three parts was considered and an analytical solution was obtained for the structure of a special discontinuity. Moreover, it was also shown that only one such discontinuity can exist for given m and μ.

An overview of works addressing solutions of Eq. (2) in the case of $\varphi(v)$ having a single inflection point can be found in [1].

It was shown in [6–10,12,14] that, in the case of $\varphi(v)$ having two inflection points, Eq. (2) with given m and μ can have several (arbitrarily many) solutions representing structures of special discontinuities. The study of this case is motivated by the fact that special discontinuities lead to a nonunique solution of the Riemann problem [6,7,19–22]. Examples of a Hugoniot curve having two points of inflection were given in [23,24]. Accordingly, the study of properties of solutions to model equation (2) for fluxes with two inflection points is not only of theoretical, but also of practical interest.

In [9–11,14] the stability of special discontinuities was analyzed using the Evans function in the case $\varphi(v) = v^4 - v^2$, which has two points of inflection. It was found that all special discontinuities, except for one propagating at the minimum velocity and having a monotone structure, are linearly (spectrally) unstable. It was conjectured that, for $\varphi(v)$ with two inflection points, only one special discontinuity is stable. Numerical computations showed that special discontinuities that are unstable with respect to infinitesimal perturbations are also unstable with respect to finite perturbations. For stable special discontinuities such that $v(x, t) \in [v_l, v_r]$, it was found that $\varphi(v)$ has only one inflection point on the interval $[v_l, v_r]$. Note that the only appearing special discontinuity considered in [2–5] is stable and the flux function $\varphi(v)$ has a single inflection point.

When the solution for a shock structure is obtained numerically, the Evans function allows one to conclude the stability or instability of only this solution. Possibly, if the solution for the case of several special discontinuities is found in analytical form, it would be possible to prove the stability or instability of special discontinuities. In this paper, such an analytical expression is obtained for $\varphi(v)$ defined as a piecewise linear function consisting of four parts such that the traveling wave solutions include the structures of several special discontinuities. An example of such a function is presented in Fig. 1. If, for any x and t, a shock structure satisfies the condition $v(x, t) > v_A$ or $v(x, t) < v_B$ (where v_A and v_B are the respective values of v at the points A and B in Fig. 1), then the solution is constructed using only three parts of $\varphi(v)$ represented in Fig. 1 and coincides with the solution found in [5].

This paper is organized as follows. In Section 2, an analytical solution for a discontinuity with a stationary structure is obtained using a piecewise linear flux function consisting of four parts. In Section 3, we describe the solution corresponding to shock waves and the first special discontinuity. Note that a piecewise linear flux function consisting of three parts was considered in [5]. In that case, only one special discontinuity can exist. The solution for it is the same as for the first special discontinuity obtained in this work. In Section 4, we show that new special discontinuities emerge as the dispersion coefficient increases (in the case of viscosity). They appear in pairs and have a nonmonotone structure. As the dispersion coefficient increases further, one of these discontinuities disappears. The main results concerning the properties of the solution are proved in the Appendix A.

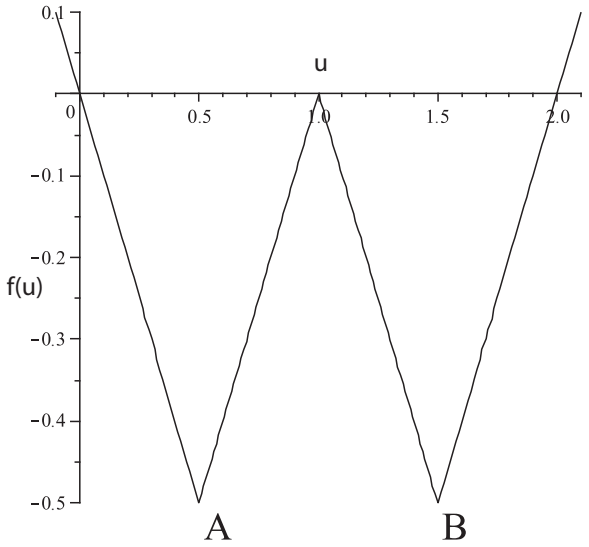


Fig. 1. $f(u)$.

2. Discontinuities with a stationary structure

In the case of large-scale phenomena, both terms on the right-hand side become small as compared with the left-hand-side terms. When the right-hand side is neglected, Eq. (2) becomes (1).

Eq. (1) admits discontinuous solutions. Eq. (1) (like (2)) represents a conservation law. Therefore, the corresponding relation across a discontinuity (shock) can be written as

$$W = \frac{[\varphi(v)]}{[v]}.$$

Here, W is the shock velocity and the difference between the values of a function behind and ahead of the shock is denoted by square brackets.

For $\mu \neq 0$, making the substitutions

$$t \to t\sqrt{m}, \quad x \to x\sqrt{m}, \quad \gamma = \sqrt{m}/\mu$$

in Eq. (2) yields

$$\frac{\partial v}{\partial t} + \frac{\partial \varphi(v)}{\partial x} + \frac{\partial^3 v}{\partial x^3} = \frac{1}{\gamma}\frac{\partial^2 v}{\partial x^2}.$$

The stationary shock structure problem is stated as follows:

$$v = v(\xi), \qquad \xi = x - Wt,$$

$$\frac{d^2 v}{d\xi^2} - \frac{1}{\gamma}\frac{dv}{d\xi} = Wv - \varphi(v), \tag{3}$$

$$\lim_{\xi \to -\infty} v(\xi) = v_l, \quad \lim_{\xi \to +\infty} v(\xi) = v_r.$$

On introducing

$$\varphi(v) = f(u - v_r), \quad v = u - v_r, \quad u_l = v_l - v_r,$$

we write problem (3) in the form

$$u = u(\xi), \qquad \xi = x - Wt,$$

$$\frac{d^2 u}{d\xi^2} - \frac{1}{\gamma}\frac{du}{d\xi} = Wu - f(u), \tag{4}$$

$$\lim_{\xi \to -\infty} u(\xi) = u_l, \quad \lim_{\xi \to +\infty} u(\xi) = 0.$$

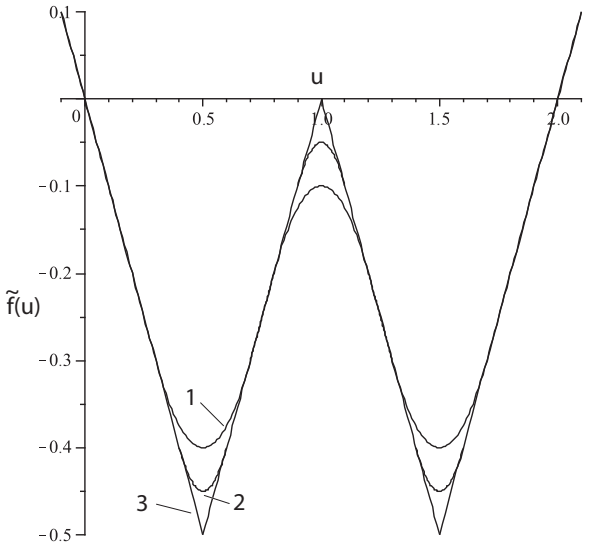


Fig. 2. $\widetilde{f}(u)$: (1) $d = 0.2$, (2) $d = 0.1$, and (3) $d = 0$.

If problem (4) is solvable, then, from a point of view of large length scales, its solution must represent a discontinuity with a stationary structure such that $u = u_l$ and $u = 0$ are the states behind and ahead of the discontinuity, i.e., for $(x < 0)$ and $(x > 0)$, respectively.

The states u_l and $u = 0$ satisfy the relations across the discontinuity

$$f(u) = Wu. \tag{5}$$

It was shown in [6,7] that Eq. (1) can have solutions representing special discontinuities if the graph of $\varphi(v)$ has two points of inflection. In [7–10,12,20,21] solutions of problem (4) with the fixed state $u = 0$ ahead of the discontinuity were found numerically depending on the parameter γ and the state u_l behind the discontinuity in the case $f(u) = (u-1)^4 - (u-1)^2$.

In this work, to obtain an analytical solution of problem (4), we define $f(u)$ as a piecewise linear function, specifically,

$$f(u) = \begin{cases} -u & \text{if } u \le 0.5 \\ u - 1 & \text{if } 0.5 < u \le 1 \\ -u + 1 & \text{if } 1 < u \le 1.5 \\ u - 2 & \text{if } 1.5 < u \end{cases} \tag{6}$$

The plot of $f(u)$ is shown in Fig. 1.

On the set $\overline{\mathbb{R}} = \{(u, d) | 0 \le d < 0.25\}$ we introduce the function

$$\widetilde{f}(u,d) = \begin{cases} -u & \text{if } u < 0.5 - d,\ d \in (0, 0.25) \\ 0.5(-1 + d + (u-0.5)^2 d^{-1}) & \text{if } 0.5 - d \le u \le 0.5 + d,\ d \in (0, 0.25) \\ -0.5 & \text{if } u = 0.5,\ d = 0 \\ u - 1 & \text{if } 0.5 + d < u < 1 - d,\ d \in (0, 0.25) \\ -0.5(d + (u-1)^2 d^{-1}) & \text{if } 1 - d \le u \le 1 + d,\ d \in (0, 0.25) \\ 0 & \text{if } u = 1,\ d = 0 \\ -u + 1 & \text{if } 1 + d < u < 1.5 - d,\ d \in (0, 0.25) \\ 0.5(-1 + d + (u-1.5)^2 d^{-1}) & \text{if } 1.5 - d \le u \le 1.5 + d,\ d \in (0, 0.25) \\ -0.5 & \text{if } u = 1.5,\ d = 0 \\ u - 2 & \text{if } 1.5 + d < u,\ d \in (0, 0.25) \end{cases} \tag{7}$$

The plot of $\widetilde{f}(u)$ for $d = 0.1$, $d = 0.5$, and $d = 0$ is presented in Fig. 2.

The properties of $\widetilde{f}(u)$ are stated in the form of Lemmas 2.1 and 2.2. Their proofs are given in the Appendix A.

Lemma 2.1. *The function $\widetilde{f}(u, d)$ is continuous with respect to u and d on the set $\overline{\mathbb{R}}$.*

Lemma 2.2. *On $\overline{\mathbb{R}}$, $|\widetilde{f}(u_2, d) - \widetilde{f}(u_1, d)| \le |u_2 - u_1|$.*

The function $\widetilde{f}(u, d)$ is continuous and satisfies

$$\lim_{d \to 0} \widetilde{f}(u, d) = \widetilde{f}(u, 0) = f(u).$$

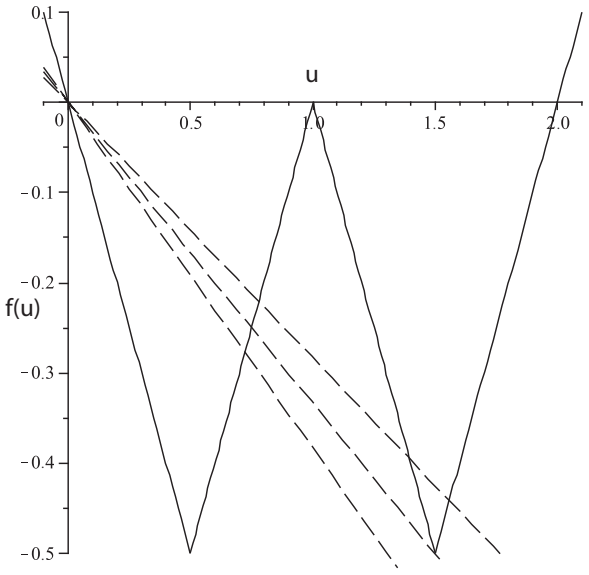


Fig. 3. $f(u)$.

Let $\widetilde{u}(\xi)$ be the solution of problem (4), i.e.,

$$\frac{d^2\widetilde{u}}{d\xi^2} - \frac{1}{\gamma}\frac{d\widetilde{u}}{d\xi} = W\widetilde{u} - \widetilde{f}(\widetilde{u}, d),$$
$$\lim_{\xi\to-\infty}\widetilde{u}(\xi) = u_l, \quad \lim_{\xi\to\infty}\widetilde{u}(\xi) = 0. \tag{8}$$

If $f(u)$ in Eq. (4) is given by (6), then

$$\lim_{d\to 0}\widetilde{u}(\xi, d) = u(\xi), \tag{9}$$

where $u(\xi)$ is the solution of problem (4).

Lemma 2.3. *The solution $\widetilde{u}(\xi, d)$ of problem (8) with $\widetilde{f}(u, d)$ given by (7) converges, as $d \to 0$, to the solution of problem (4) with $f(u)$ given by (6), i.e., (9) holds. The proof of Lemma 2.3 is given in the Appendix A.*

By using Lemmas 2.1–2.3, we will study the structure problem in the case of $f(u)$ given by (6). Then the shock structure equation becomes

$$\frac{d^2u}{d\xi^2} - \frac{1}{\gamma}\frac{du}{d\xi} = \begin{cases} (W+1)u & \text{if } u \le 0.5 \\ (W-1)u+1 & \text{if } 0.5 < u \le 1 \\ (W+1)u-1 & \text{if } 1 < u \le 1.5 \\ (W-1)u+2 & \text{if } 1.5 < u \end{cases} \tag{10}$$

It follows from (5) that the minimum $W = -1$ is attained at $u_l = 0.5$. For $f(u)$ defined by (6), admissible are the values $W \in (-1, 1)$. The same value of W can be associated with different states u_l (behind the shock), which are the roots of Eq. (5). This equation has two roots for $W \in (-1, -1/3)$, three roots for $W = -1/3$, and four roots for $W \in (-1/3, 0)$ (see Fig. 3).

Let $U_1 = 0.5$, $U_2 = 1$, and $U_3 = 1.5$. The intervals S_n, $n = 1, \ldots, 4$, are defined as $S_1 = \{-\infty < u < U_1\}$, $S_2 = \{U_1 < u < U_2\}$, $S_3 = \{U_2 < u < U_3\}$, and $S_4 = \{U_3 < u < \infty\}$.

Let ξ_k be a sequence of discrete values of ξ such that $u(\xi_k) = U_j$, where $j = 1, 2, 3$, $\xi_1 = 0$, $\xi_{k+1} < \xi_k$, and $u(\xi) \neq U_j$ for any $\xi \in (\xi_{k+1}, \xi_k)$. An example of $\{\xi_k\}$ is given in Fig. 4, which shows the graph of the solution $u(\xi)$ to problem (4) for parameter values $\gamma = 9.5$ and $W = -0.136$. As ξ decreases, it sequentially takes the values $\xi_1 \Rightarrow \xi_2 \Rightarrow \xi_3 \Rightarrow \xi_4 \Rightarrow \xi_5 \Rightarrow \xi_6$ and $u(\xi)$ belongs to S_1, S_2, S_3, S_4, S_3, S_2, and S_3, respectively (in what follows, we use the notation $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_3$).

The finite sequence $\{\xi_k\}$ thus constructed includes all points at which the solution has the values $\{U_j\}$. Here, $k = 1, 2, \ldots, K$, where K is determined by the parameter γ [6–8].

To determine which of the domains S_n, $n = 1, \ldots, 4$, contains $u(\xi)$ for $\xi \in (\xi_{k+1}, \xi_k)$, it is sufficient to know the value $u(\xi_k)$ and its derivative $\left.\frac{du}{d\xi}\right|_{\xi=\xi_k}$ at the point $\xi = \xi_k$. The value $u(\xi_k)$ determines the boundary between S_n that contains the solution at $\xi = \xi_k$. For example, if $u(\xi_k) = U_1 = 0.5$, this is the boundary between S_1 and S_2. Moreover, as ξ decreases,

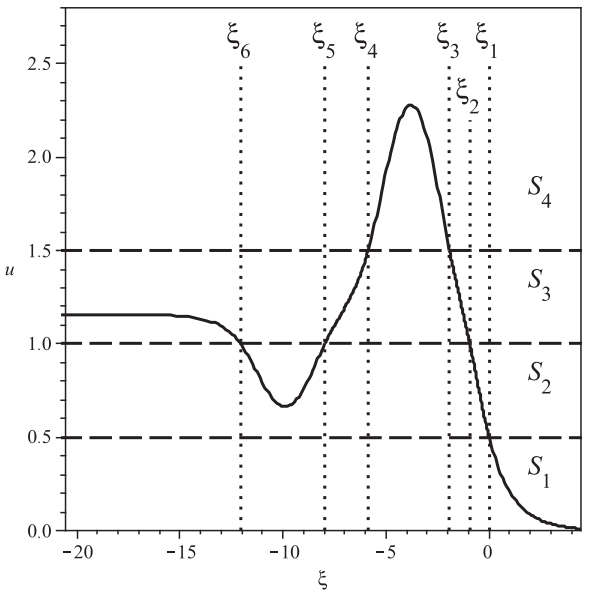


Fig. 4. Shock structure for $\gamma = 9.5$ and $W = -0.136$. The sequence $\{\xi_k\}$, $k = 1, 2, \ldots 6$, and the domains S_1, ..., S_4 are shown.

the solution moves into S_2 if $\left.\frac{du}{d\xi}\right|_{\xi=\xi_k} < 0$ and remains in S_1 otherwise. Denote by $u_{nk}(\xi)$ the function $u(\xi)$ defined on the intervals (ξ_k, ξ_{k-1}) with values in S_n. Then

$$u_{nk} = \begin{cases} u_{1k}(\xi) \in S_1 & \text{if } (\xi_{k-1} = U_1 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} > 0); \\ u_{2k}(\xi) \in S_2 & \text{if } (\xi_{k-1} = U_1 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} < 0) \vee (\xi_k = U_2 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} > 0); \\ u_{3k}(\xi) \in S_3 & \text{if } (\xi_{k-1} = U_2 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} < 0) \vee (\xi_{k-1} = U_3 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} > 0); \\ u_{4k}(\xi) \in S_4 & \text{if } (\xi_{k-1} = U_3 \wedge \left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} < 0). \end{cases} \tag{11}$$

With the notation introduced, problem (4) can be rewritten as

$$\frac{d^2u_{11}}{d\xi^2} - \frac{1}{\gamma}\frac{du_{11}}{d\xi} = Wu_{11} - f(u_{11}),\ n = 1, k = 1,\ \xi \in [\xi_1, \infty)$$
$$u_{11}(\xi_1) = U_1 = 0.5,\ \lim_{\xi\to+\infty} u_{11}(\xi) = 0 \tag{12}$$

$$\frac{d^2u_{nk}}{d\xi^2} - \frac{1}{\gamma}\frac{du_{nk}}{d\xi} = Wu_{nk} - f(u_{nk}),\ \xi \in [\xi_k, \xi_{k-1})$$
$$u_{nk}(\xi_{k-1}) = u_{n,k-1}(\xi_{k-1}),$$
$$\left.\frac{du_{nk}}{d\xi}\right|_{\xi=\xi_{k-1}} = \left.\frac{du_{n,k-1}}{d\xi}\right|_{\xi=\xi_{k-1}},\ k = 2, \ldots, K,\ n = 1, \ldots, 4 \tag{13}$$

If $\xi \geq \xi_1$ problem (12) has the solution

$$u_{11}(\xi) = U_1 e^{(\frac{1-\sqrt{1+4\gamma^2(W+1)}}{2\gamma})\xi}.$$

The solution of problem (13) is represented in the form

$$u_{nk}(\xi) = A_{nk}e^{\alpha_{nk}\xi} + B_{nk}e^{\beta_{nk}\xi} + C_{nk},\quad k = 2, \ldots, K. \tag{14}$$

Here, the index n determines the domain S_n, while the index k determines the interval (ξ_k, ξ_{k-1}).
Substituting (14) into (13) and taking into account (10), we obtain

$$u_{nk}(\xi) = \begin{cases} A_{1k}e^{\sigma_{11}\xi} + B_{1k}e^{\sigma_{12}\xi}, & u_{1k}(\xi) \in S_1 \\ A_{2k}e^{\sigma_{21}\xi} + B_{2k}e^{\sigma_{22}\xi} - (W-1)^{-1}, & u_{2k}(\xi) \in S_2 \\ A_{3k}e^{\sigma_{11}\xi} + B_{3k}e^{\sigma_{12}\xi} + (W+1)^{-1}, & u_{3k}(\xi) \in S_3 \\ A_{4k}e^{\sigma_{21}\xi} + B_{4k}e^{\sigma_{22}\xi} - 2(W-1)^{-1}, & u_{4k}(\xi) \in S_4 \end{cases} \tag{15}$$

where

$$\alpha_{1k} = \alpha_{3k} = \sigma_{11} = \frac{1+\sqrt{D_1}}{2\gamma} \tag{16}$$

$$\beta_{1k} = \beta_{3k} = \sigma_{12} = \frac{1-\sqrt{D_1}}{2\gamma} \tag{17}$$

$$\alpha_{2k} = \alpha_{4k} = \sigma_{21} = \frac{1+\sqrt{D_2}}{2\gamma}$$

$$\beta_{2k} = \beta_{4k} = \sigma_{22} = \frac{1-\sqrt{D_2}}{2\gamma}$$

$$D_1 = 1 + 4\gamma^2(W+1), \qquad D_2 = 1 + 4\gamma^2(W-1) \tag{18}$$

$$C_{1k} = 0, \quad C_{2k} = -(W-1)^{-1}, \quad C_{3k} = (W+1)^{-1}, \quad C_{4k} = -2(W-1)^{-1}$$

The coefficients A_{nk}, B_{nk} for $k>1$ and $n=1,\dots,4$ in (15) can be found using the following recurrences, which ensure the continuity of $u(\xi)$ and its derivative $du/d\xi$ at $\xi=\xi_{k-1}$:

$$u_{nk}(\xi_{k-1}) = u_{n,k-1}(\xi_{k-1}) \tag{19}$$

$$\left.\frac{du_{nk}}{d\xi}\right|_{\xi=\xi_{k-1}} = \left.\frac{du_{n,k-1}}{d\xi}\right|_{\xi=\xi_{k-1}} \tag{20}$$

The recurrence procedure for determining $u_{nk}(\xi)$ is constructed as follows. Given $u_{n,k-1}(\xi_{k-1})$ and $du_{n,k-1}/d\xi$ at $\xi=\xi_{k-1}$, conditions (11) are used to determine which of the domains S_n contains $u_{nk}(\xi)$. Next, the solution in this domain is determined using the corresponding equality in (15).

Now we determine the value of $u_{nk}(\xi)$ at $\xi=\xi_k$. If $u_{nk}(\xi)\in S_1$ or $u_{nk}(\xi)\in S_4$, then ξ_k is found as a value satisfying the condition $\xi_k<\xi_{k-1}$ and being the largest root of the equation $u_{nk}(\xi_k)=U_1$ or $u_{nk}(\xi_k)=U_3$, respectively. If $u_{nk}(\xi)\in S_2$ or $u_{nk}(\xi)\in S_3$, we find ξ_k satisfying $\xi_k<\xi_{k-1}$ and being the largest root of the equation $u_{nk}(\xi_k)=u_{nk}(\xi_{k-1})$ or $u_{nk}(\xi_k)=U_j$, where U_j is the value corresponding to the boundary of S_n and $U_j\neq u_{nk}(\xi_{k-1})$. For example, for the solution shown in Fig. 4, the values of $u(\xi)$ at $\xi\in(\xi_5,\xi_4)$ belong to S_3 and the equalities $u_{34}(\xi_4)=U_3=1.5$ and $u_{34}(\xi_5)=U_2=1$ hold on the boundary of S_3.

This completes the construction of the solution $u_{nk}(\xi)$.

3. Conditions for the existence of a special discontinuity

At a special discontinuity, an additional relation [6–8,15,16,25] ensuring its existence has to be satisfied. It was shown in [6,7] that numerous special discontinuities can exist when γ is sufficiently large. Let us find the condition for the existence of the first special discontinuity.

If the solution satisfies

$$u(\xi) < 3/2 \tag{21}$$

for any ξ, then it can be constructed using only the first three parts of the piecewise linear function (6), i.e., we can use the function

$$f_1(u) = \begin{cases} -u & \text{if } u \le 0.5 \\ u-1 & \text{if } 0.5 < u \le 1 \\ -u+1 & \text{if } 1 < u \end{cases}$$

Then, by applying a change of variables, the problem can be reduced to

$$\gamma_a \frac{d^2u}{d\xi^2} = -\frac{du}{d\xi} + f_2(u) - f_2(u_-) - W(u-u_-), \qquad \gamma_a = \frac{1}{\gamma}, \tag{22}$$

where

$$f_2(u) = \begin{cases} u & \text{if } u \le 0 \\ 0 & \text{if } 0 < u \le 1 \\ b(u-1)u+2 & \text{if } u \ge 1; \end{cases}$$

this problem was solved in [5]. Here, b is a parameter and u_- is the state behind the discontinuity.

In this section, we consider solutions satisfying condition (21). Therefore, they can be obtained from the solutions found in Section 3 in [5]. For this reason, Lemmas 3.1 and 3.2 and Propositions 3.1 and 3.2 are given without proofs, since they

are restatements or consequences of the results proved in Section 3 in [5]. Used in this paper, the shock structure equation (4) is equivalent to Eq. (22), but differs from the latter in form and notation. This form is important to retain in order to compare the results with those obtained in [7–10,12,20,21]. As will be shown below, condition (21) holds only for the first special discontinuity. Accordingly, the other ones cannot be described by the solutions presented in [5] and are obtained here for the first time.

The first special discontinuity propagates at the lowest velocity W_0 (highest in absolute value) among all possible special discontinuities [7]. The solution for the shock structure is constructed by applying the recurrence procedure described above.

For $\xi_2 < \xi < \xi_1 = 0$ (see (15)), the solution has the form

$$u_{22}(\xi) = A_{22}e^{\sigma_{21}\xi} + B_{22}e^{\sigma_{22}\xi} - (W-1)^{-1}, \quad u_{22}(\xi) \in S_2$$

The coefficients A_{22} and B_{22} are determined using system (19), (20). For $u_{22}(\xi)$, it becomes

$$A_{22} + B_{22} - (W-1)^{-1} = U_1$$
$$A_{22}\sigma_{21} + B_{22}\sigma_{22} = U_1\sigma_{12}$$

Therefore,

$$A_{22} = \frac{2 - \sqrt{D_1} - \sqrt{D_2} + W\left(\sqrt{D_1} - \sqrt{D_2}\right)}{4\sqrt{D_2}(1-W)}$$

$$B_{22} = \frac{2 - \sqrt{D_1} + \sqrt{D_2} + W\left(\sqrt{D_1} + \sqrt{D_2}\right)}{4\sqrt{D_2}(W-1)}$$

The second equality in (18) implies that $0 < D_2 < 1$ for $1 - 1/(4\gamma^2) < W < 1$. In this case, σ_{21} and σ_{22} are real (since $D_2 > 0$) and positive (since $1 - \sqrt{D_2} > 0$).

The condition $1 - 1/(4\gamma^2) < W$ is satisfied for any $W \in (-1, 1)$ if $\gamma < 1/(2\sqrt{2})$ and is violated for any $W \in (-1, 0)$ if $\gamma > 1/2$. Therefore, σ_{21} and σ_{22} are positive for $\gamma < 1/(2\sqrt{2})$ and complex for $\gamma > 1/2$.

Lemma 3.1. *If $\xi \le \xi_1 = 0 \wedge \gamma < 1/(2\sqrt{2}) \wedge W \in (-1, 0)$, then $u_{22}(\xi) \le U_2$ and $du_{22}(\xi)/d\xi < 0$.*

Under the conditions of Lemma 3.1, the solution represents an integral curve in the phase plane that joins two special points: a saddle (the state ahead of the discontinuity) and a node (the state behind the discontinuity).

According to (15), when ξ_2 is such that $u_{22}(\xi_2) = U_2 = 1$, the analytical expression for $u(\xi)$ changes.

By Lemma 3.1, U_2 is never reached if $\gamma < 1/(2\sqrt{2})$. In this case, σ_{21} and σ_{22} are positive and the function $u_{22}(\xi)$ varies monotonically from U_1 at $\xi = 0$ to $1/(1-W)$ as $\xi \to -\infty$.

For $1 - 1/(4\gamma^2) \le W \le 0$, the solution of structure problem (4) has the form

$$u(\xi) = \begin{cases} U_1 e^{\sigma_{12}\xi} & \text{if } \xi \ge \xi_1 = 0 \\ A_{22}e^{\sigma_{21}\xi} + B_{22}e^{\sigma_{22}\xi} - (W-1)^{-1} & \text{if } \xi_2 < \xi < \xi_1 = 0 \end{cases} \tag{23}$$

where σ_{21} and σ_{22} are positive real numbers.

An example of a shock structure obtained using formulas (23) is given in Fig. 5. In the phase plane, this solution represents an integral curve joining two special points: a saddle (the state ahead of the discontinuity) and a node (the state behind the discontinuity).

In the case of complex σ_{21} and σ_{22}, i.e., for $W < 1 - 1/(4\gamma^2)$, it is more convenient to write the solution as

$$u_{22}(\xi) = \frac{e^{\frac{\xi}{2\gamma}}}{2(W-1)}\left[\sqrt{-D_2{}^{-1}}(2 + (W-1)\sqrt{D_1})\sin\left(\frac{\sqrt{-D_2}\,\xi}{2\gamma}\right)\right.$$
$$\left. + (W+1)\cos\left(\frac{\sqrt{-D_2}\,\xi}{2\gamma}\right)\right] - \frac{1}{W-1}.$$

If $u_{22}(\xi) < 1$ for $\xi < \xi_1$, then the state behind the discontinuity corresponds to a focus-type special point (Fig. 6).

The maximum value of $u_{22}(\xi)$ is reached at

$$\xi_{max}(W, \gamma) = \frac{2\gamma}{\sqrt{-D_2}}\left(\arctan\left(\frac{\sqrt{-D_2}}{\sqrt{D_1}}\right) - \pi\right)$$

and is given by

$$u_{max}(W, \gamma) = \frac{1}{1-W}\left(1 + \frac{\sqrt{2}}{4\gamma}\left(\sqrt{D_1} - 1\right)e^{\left(-\pi + \arctan\left(\frac{\sqrt{-D_2}}{\sqrt{D_1}}\right)\right)\frac{1}{\sqrt{-D_2}}}\right) \tag{24}$$

Suppose that, at $W = W^*$, we have $u_{max}(W^*, \gamma) = 1$. Then it follows from (24) that $W^*(0.5) = 0$ and $W^* \to -0.5$ as $\gamma \to \infty$. The plot of W^* as a function of γ is shown in Fig. 7.

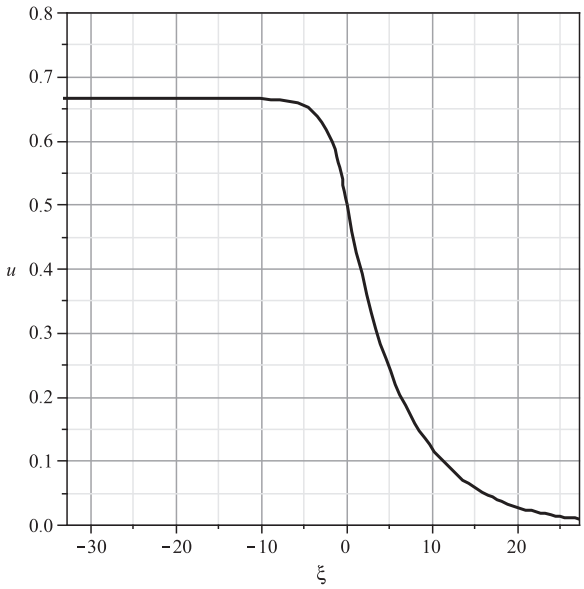


Fig. 5. Shock structure for $\gamma = 0.3 < 1/(2\sqrt{2})$ and $W = -0.5$.

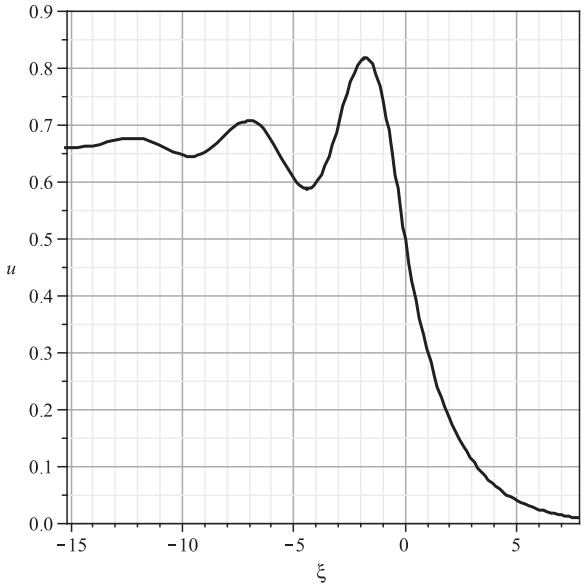


Fig. 6. Shock structure for $\gamma = 2$ and $W = -0.5$.

Lemma 3.2. *$u_{max}(W, \gamma) < 1$ if $0 < \gamma < 0.5$ and $-1 < W < 0$.*

Let ξ_2 be the largest root (recall that $\xi < 0$) of the equation

$$U_2 = u_{22}(\xi) \tag{25}$$

or

$$U_2 = A_{22}e^{\sigma_{21}\xi} + B_{22}e^{\sigma_{22}\xi} - (W-1)^{-1}. \tag{26}$$

Then, for $\xi < \xi_2$, solution (15) has the form

$$u_{33}(\xi) = A_{33}e^{\sigma_{11}\xi} + B_{33}e^{\sigma_{12}\xi} + (W+1)^{-1} \tag{27}$$

According to Lemma 3.2, Eq. (25) has a solution only if $\gamma > 0.5$. This condition is necessary and, as will be shown below, sufficient for the existence of special discontinuities.

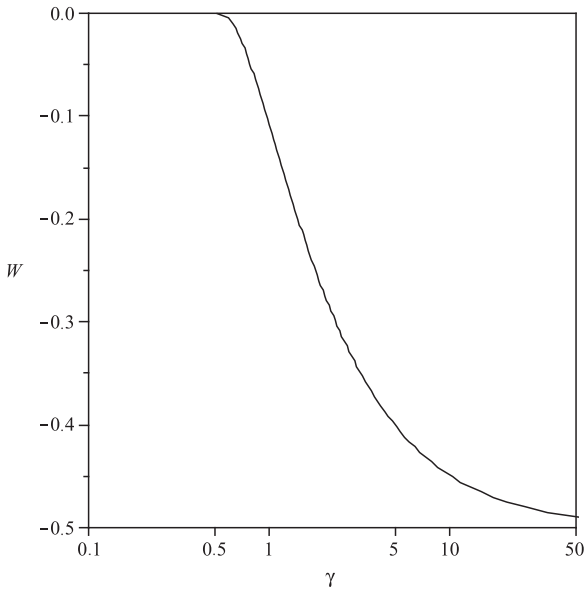


Fig. 7. W^* as a function of γ; $u_{max}(W^*, \gamma) = 1$.

The coefficients A_{33} and B_{33} can be found using conditions (19) and (20) for the continuity of $u(\xi)$ and its derivative at $\xi = \xi_2$:

$$U_2 = A_{33}e^{\sigma_{11}\xi_2} + B_{33}e^{\sigma_{12}\xi_2} + (W+1)^{-1}, \tag{28}$$

$$A_{22}\sigma_{21}e^{\sigma_{21}\xi_2} + B_{22}\sigma_{22}e^{\sigma_{22}\xi_2}, = A_{33}\sigma_{11}e^{\sigma_{11}\xi_2} + B_{33}\sigma_{12}e^{\sigma_{12}\xi_2} \tag{29}$$

where σ_{11} and σ_{12} are real numbers of opposite signs (see (16)–(18)).

If $B_{33} = 0$, then the constructed solution (27) remains in S_3 for $\xi \in (-\infty, \xi_2)$ and the integral curve joins two special points of saddle type. In this case, Eqs. (28) and (29) yield

$$U_2 = A_{33}e^{\sigma_{11}\xi_2} + (W+1)^{-1} \tag{30}$$

$$A_{22}\sigma_{21}e^{\sigma_{21}\xi_2} + B_{22}\sigma_{22}e^{\sigma_{22}\xi_2} = A_{33}\sigma_{11}e^{\sigma_{11}\xi_2} \tag{31}$$

Eliminating A_{33} from system (30), (31) gives

$$A_{22}\sigma_{21}e^{\sigma_{21}\xi_2} + B_{22}\sigma_{22}e^{\sigma_{22}\xi_2} = (U_2 - (W+1)^{-1})\sigma_{11}, \tag{32}$$

where σ_{21}, σ_{22}, A_{22}, and B_{22} are explicit functions of W and γ, while ξ_2 is an implicit function of W and γ, which is determined by Eq. (28). Thus, (32) relates W and γ at the first special discontinuity.

Equality (30) is used to determine A_{33}. Since $B_{33} = 0$, we derive the expression

$$u_{33}(\xi) = (U_2 - (W+1)^{-1})e^{\sigma_{11}(\xi-\xi_2)} + (W+1)^{-1}.$$

Accordingly,

$$u(\xi) = \begin{cases} U_1 e^{\sigma_{12}\xi} & \text{if } \xi \geq \xi_1 = 0 \\ A_{22}e^{\sigma_{21}\xi} + B_{22}e^{\sigma_{22}\xi} - (W-1)^{-1} & \text{if } \xi_2 \leq \xi < \xi_1 \\ (U_2 - (W+1)^{-1})e^{\sigma_{11}(\xi-\xi_2)} + (W+1)^{-1} & \text{if } \xi < \xi_2. \end{cases} \tag{33}$$

This completes the construction of analytical solution (33) for the first special discontinuity. Moreover, we have obtained an additional condition for the existence of a structure in this special discontinuity (equality (32)) and the following result has been proved.

Proposition 3.1. *For the shock structure problem (4) in the case of $f(u)$ defined by (6) with the state $u = 0$ ahead of the discontinuity and a state u_l from the interval (U_2, U_3) behind the discontinuity, the solution is a structure of a special discontinuity if equality (32) holds.*

Equality (32) is the condition for the existence of a structure of a special discontinuity and is an additional condition at this discontinuity that follows from the requirement for the existence of a stationary structure.

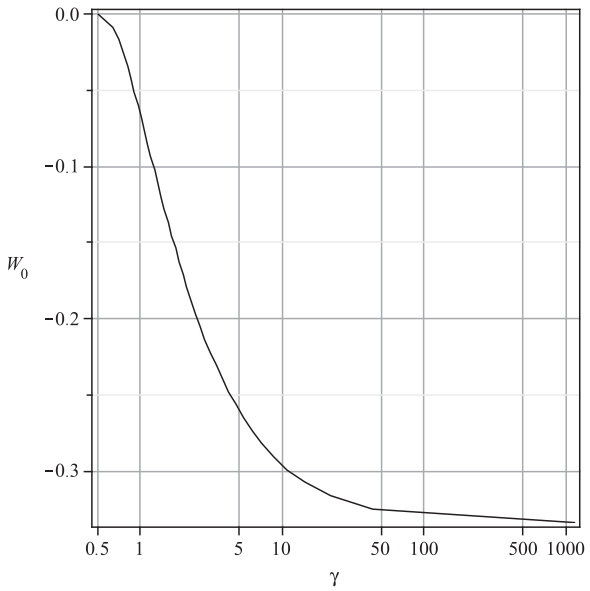


Fig. 8. Velocity of the first special discontinuity W_0 against γ.

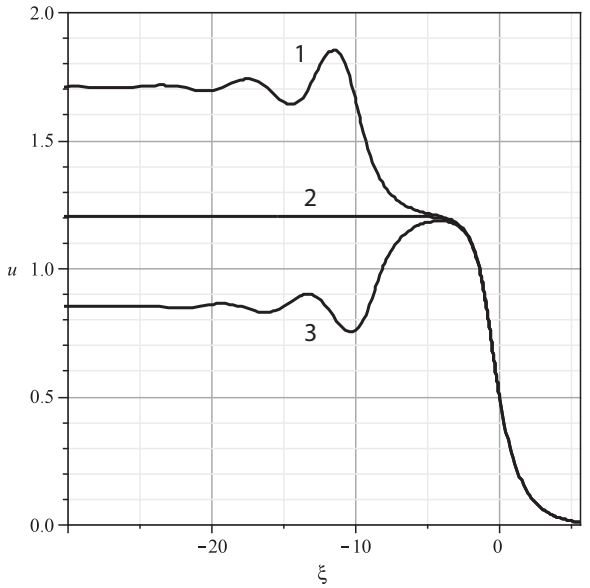


Fig. 9. Shock structure for $\gamma \approx 2.006$: (1) $W_0 \approx -0.1707$, (2) $W = -0.17$, and (3) $W = -0.172$.

Note that W and ξ_2 in (33) are implicit functions of γ, which are determined by Eqs. (26) and (32). If γ is given, W and ξ_2 are determined by numerically solving algebraic equations (26) and (32) and the results are then substituted into Eqs. (33).

Fig. 8 shows the velocity W_0 of the first special discontinuity as a function of γ.

Proposition 3.2. *For $\gamma \le 0.5$, there are no special discontinuities. For $\gamma > 0.5$, there is at least one special discontinuity. It has a monotone structure and a velocity $W \in (-1/3, 0)$ with $W \to 0$ as $\gamma \to 0.5$ and $W \to -1/3$ as $\gamma \to \infty$.*

In [10] in the case $\varphi(u) = u^4 - u^2$, no special discontinuities were found in numerical computations for $\gamma \le 1.19$ and it was noted that, depending on the form of $\varphi(u)$, there is a minimum value of the dispersion coefficient at which special discontinuities appear. In [5] this fact was proved for a piecewise linear flux function consisting of three parts.

An example of a shock structure for a special discontinuity C_0 for $\gamma \approx 2.006$ and $W_0 \approx -0.1707$ is shown in Fig. 9. Additionally, the figure presents shock structures with velocities slightly different from W_0 (-0.17 and -0.172). Both discontinuities have an oscillatory structure and correspond to a focus-type special point. The corresponding expression for the analyt-

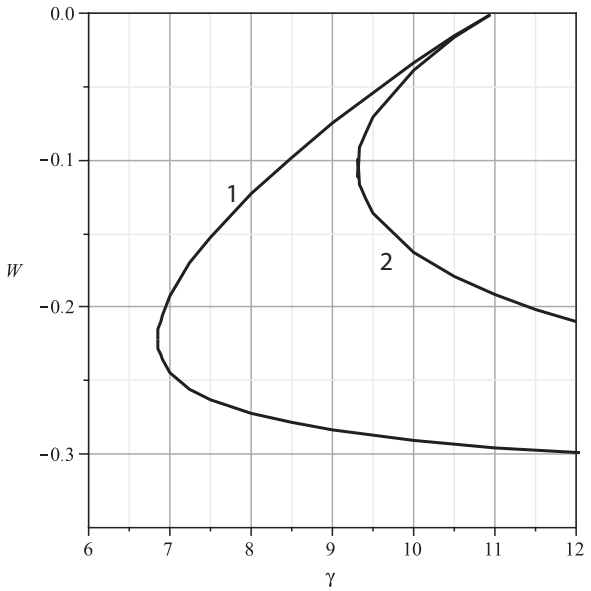


Fig. 10. Velocity of a special discontinuity as a function of γ: (1) the first pair of special discontinuities and (2) the second pair of special discontinuities.

ical solution in (15) is used for each of the domains S_1,..., S_4. For $W = -0.17$ (curve 1), the solution sequentially belongs to S_1, S_2, S_3, and S_4 ($S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4$). In the case of the special discontinuity in curve 2, the solution does not reach S_4 and sequentially passes $S_1 \Rightarrow S_2 \Rightarrow S_3$. For $W = -0.172$ (curve 3), the shock structure is associated with the sequence $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_2$).

4. Features concerning the appearance of the second special discontinuity

If $W > W_0$, where W_0 is the velocity of the first special discontinuity, then the solution at ξ_3 reaches the value $U_3 = 1.5$. This solution has the form of (27) for $\xi \in (\xi_3, \xi_2)$. The coefficients A_{33} and B_{33} are found by solving system (28), (29), while ξ_3 is determined by the equation $u_{33}(\xi_3) = U_3$.

For $\xi \in (\xi_4, \xi_3)$,

$$u_{44}(\xi) = A_{44}e^{\sigma_{21}\xi} + B_{44}e^{\sigma_{22}\xi} - 2(W-1)^{-1}.$$

The coefficients A_{44} and B_{44} are determined by system (19), (20), which becomes

$$U_3 = A_{44}e^{\sigma_{21}\xi_3} + B_{44}e^{\sigma_{22}\xi_3} - 2(W-1)^{-1}$$

$$A_{44}\sigma_{21}e^{\sigma_{21}\xi_3} + B_{44}\sigma_{22}e^{\sigma_{22}\xi_3} = A_{33}\sigma_{11}e^{\sigma_{11}\xi_3} + B_{33}\sigma_{12}e^{\sigma_{12}\xi_3}$$

Let ξ_4 be the second largest (in absolute value) root (after ξ_3) of the equation

$$A_{44}e^{\sigma_{21}\xi_4} + B_{44}e^{\sigma_{22}\xi_4} - 2(W-1)^{-1} = U_3$$

For $\xi < \xi_4$, the solution is given by the formula

$$u_{35}(\xi) = A_{35}e^{\sigma_{11}\xi} + B_{35}e^{\sigma_{12}\xi} + (W+1)^{-1}$$

As was noted above, $\sigma_{11} > 0$ and $\sigma_{12} < 0$. Therefore, the constructed solution joins two saddle-type special points if $B_{35} = 0$ (only in this case the solution is bounded as $\xi \to -\infty$). Then the smoothness conditions (19) and (20) for the function $u(\xi)$ yield the following condition for the existence of the second special discontinuity:

$$A_{35}e^{\sigma_{11}\xi_4} + (W+1)^{-1} = U_3 \tag{34}$$

$$A_{44}\sigma_{21}e^{\sigma_{21}\xi_4} + B_{44}\sigma_{22}e^{\sigma_{22}\xi_4} = A_{35}\sigma_{11}e^{\sigma_{11}\xi_4} \tag{35}$$

Eliminating A_{35} from system (34), (35), we obtain

$$A_{44}\sigma_{21}e^{\sigma_{21}\xi_4} + B_{44}\sigma_{22}e^{\sigma_{22}\xi_4} = \sigma_{11}(U_3 - (W+1)^{-1}) \tag{36}$$

where σ_{21}, σ_{22}, and σ_{11} are explicit functions of W and γ, while A_{44}, B_{44}, and ξ_4 are implicit functions of W and γ, which are determined by the above equations. Formula (36) relates W and γ at the second special discontinuity (is an additional relation at this discontinuity).

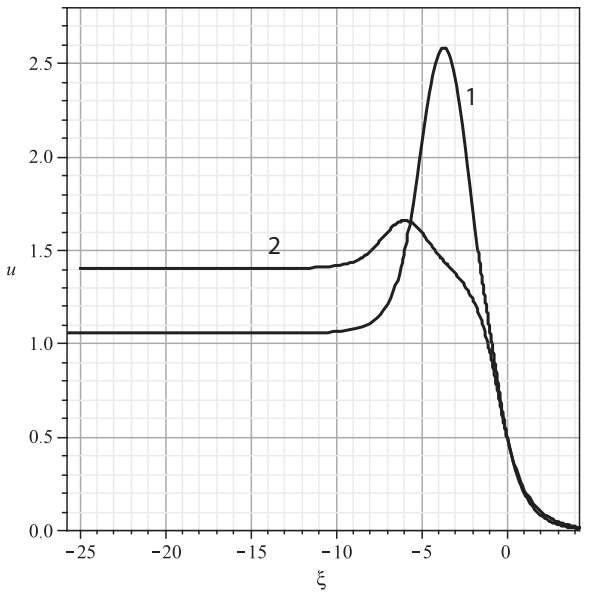


Fig. 11. Structures of special discontinuities $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3$ for $\gamma = 9.5$: (1) $W_0 \approx -0.0538$ and (2) $W_0 \approx -0.287$.

Thus, we have proved the following result.

Proposition 4.1. *The solution describing the structure of a special discontinuity for $\xi \in (-\infty, \xi_{k-1})$ in the domain S_3 has the form*

$$u_{3k}(\xi) = (u(\xi_{k-1}) - (W+1)^{-1}) e^{\sigma_{11}(\xi - \xi_{k-1})} + (W+1)^{-1}, \tag{37}$$

and the condition for the existence of a special discontinuity is given by $B_{3k} = 0$ or by the equivalent equation

$$\left.\frac{du}{d\xi}\right|_{\xi=\xi_{k-1}} = \sigma_{11}\left(u(\xi_{k-1}) - \frac{1}{W+1}\right)$$

Equality (37) describes the structure of the first monotone discontinuity for $k = 3$ and the structures of special discontinuities propagating at lower (in absolute value) velocities (for $k = 5, 7, 9\ldots$).

Eq. (36) has a unique solution if $\gamma = \gamma_1$, two solutions if $\gamma_1 < \gamma < \gamma_2$, and a single solution if $\gamma > \gamma_2$ ($\gamma_1 \approx 6.85$, $\gamma_2 \approx 10.95$). These solutions are depicted by curve 1 in Fig. 10.

In the case under consideration, special discontinuities, except for the first one, appear in pairs. Both have a similar structure, namely, the same numbers of maxima and minima and the same number of points ξ_k at which the formulas for the analytical solution change. As an example, Fig. 11 shows the structure of the pair of special discontinuities corresponding to Eq. (36) for $\gamma = 9.5$ ($S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3$).

Both structures have a single maximum, which is reached in S_4 (for $u > 3/2$). Thus, for a fixed γ, there can exist special discontinuities with the same number of extrema.

A similar algorithm is used to obtain a condition for appearing the next pair of special discontinuities as γ increases (an analytical solution for them is derived using formulas (15) in the sequence of domains $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_3$). These special discontinuities correspond to curve 2 in Fig. 10.

The second pair of special discontinuities appears at $\gamma_3 \approx 9.32$ and exists for $\gamma \in (\gamma_3, \gamma_2)$. For $\gamma > \gamma_2$, this pair degenerates into the only solution corresponding to the lower (higher in absolute value) velocity.

Fig. 10 shows that both pairs of special discontinuities exist at $\gamma = 9.5$. The structures of the second pair are presented in Fig. 12. They have a maximum (in S_4) and a minimum (in S_2).

At $\gamma = \gamma_2$, when the velocities of the special discontinuities are reduced to zero (see Fig. 10), one of them in each pair ceases to exist. Thus, the number of special discontinuities remains odd. For small values of W, the structures of the special discontinuities from the first and second pair approach each other, as can be seen in Fig. 13.

A similar scenario of appearing a pair of special discontinuities (when the shock structures have an identical number of extrema) was also observed for $\phi(u) = u^4 - u^2$ in [21]. In this case, there is no analytical solution, but the numerical solution of the corresponding differential equations shows that the first pair of special discontinuities appears at $\gamma_1 \approx 10.52$ and the profiles of $u(\xi)$ have only one extremum (maximum). These structures are presented in Fig. 14.

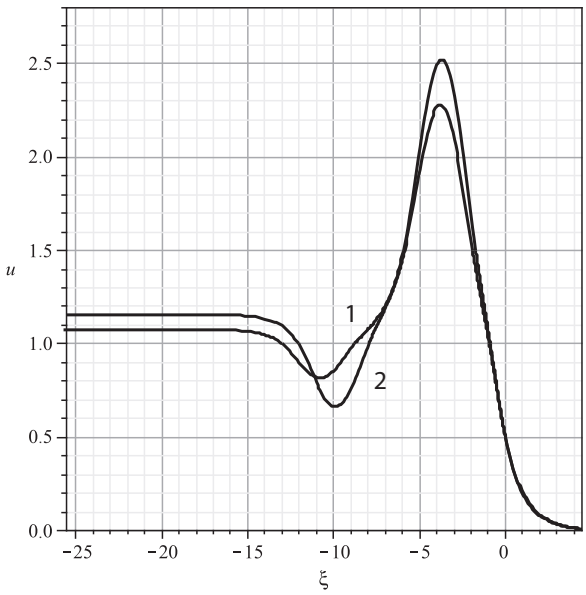


Fig. 12. Structures of special discontinuities $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_3$ for $\gamma = 9.5$: (1) $W_0 \approx -0.0707$ and (2) $W_0 \approx -0.136$.

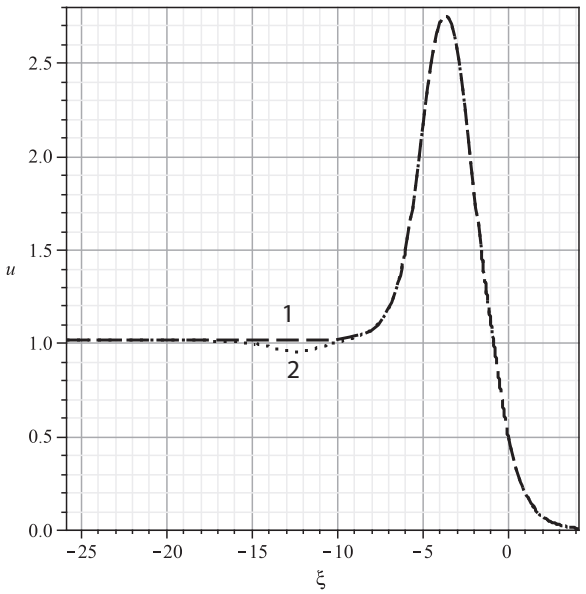


Fig. 13. Structures of special discontinuities for $\gamma = 10.5$: (1) $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3$ and (2) $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4 \Rightarrow S_3 \Rightarrow S_2 \Rightarrow S_3$.

5. Conclusions

The nonlinearity in a model generalized KdVB equation was defined as a piecewise linear function. As a result, an analytical solution of the shock structure problem was obtained and solutions representing stationary structures of special discontinuities were examined in detail. Additional conditions on these discontinuities were found. The dynamics of appearance and disappearance of special discontinuities were studied. It was shown that, as the parameter γ increases, special discontinuities, except for the first, appear in pairs and have similar structures. As γ increases further, one of the discontinuities disappears simultaneously with disappearing a discontinuity from the next pair. The total number of discontinuities decreases, but then increases again when the next pair is formed.

For a piecewise linear function consisting of four parts and for a given dissipation coefficient, it was found that there is a minimum dispersion coefficient at which a special discontinuity exists. This finding agrees with the results of Hayes and Shearer [5], where the nonlinearity was defined in such a manner that only one special discontinuity existed.

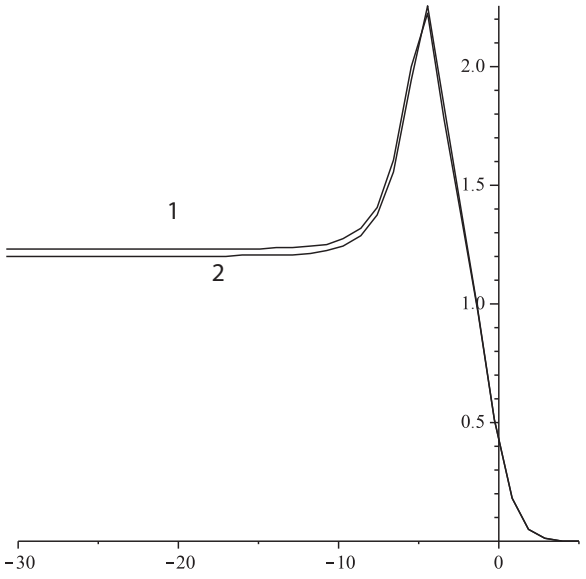


Fig. 14. Structures of special discontinuities near the point of appearance for $\gamma = 10.526$: (1) $u_l = 1.2305$ and (2) $u_l = 1.21013$.

It was shown that the piecewise linear function describing the nonlinearity is close to $\varphi(v) = v^4 - v^2$, for which the structure problem was solved numerically [6–10,12] and the properties of these solutions were found to agree with those of analytical solutions for the structures of special discontinuities obtained in this paper.

Appendix A. Appendices

Lemma 2.1. *The function $\widetilde{f}(u,d)$ is continuous with respect to u and d on the set $\overline{\mathbb{R}}$.*

Proof. Define

$$f_1(u) = -u,\ f_2(u,d) = 0.5(-1+d+(u-0.5)^2 d^{-1}),$$
$$f_3(u) = u-1,\ f_4(u,d) = -0.5(d+(u-1)^2 d^{-1}),$$
$$f_5(u) = -u+1,\ f_6(u,d) = 0.5(-1+d+(u-1.5)^2 d^{-1}),$$
$$f_7(u) = u-2.$$

Combining the continuity of $f_1(u)$, $f_2(u,\ d)$, $f_3(u)$, $f_4(u,\ d)$, $f_5(u)$, $f_6(u,\ d)$, $f_7(u)$ with the identities

$$f_1(0.5-d) \equiv f_2(0.5-d,d),\ f_2(0.5+d,d) \equiv f_3(0.5+d),$$
$$f_3(1-d) \equiv f_4(1-d,d),\ f_4(1+d,d) \equiv f_5(1+d),$$
$$f_5(1.5-d) \equiv f_6(1.5-d,d),\ f_6(1.5+d,d) \equiv f_7(1.5+d),$$

we see that $\widetilde{f}(u,d)$ is continuous for $d \neq 0$.

Since $f_1(u)$, $f_3(u)$, $f_5(u)$, and $f_7(u)$ are continuous and independent of d, we conclude that $\widetilde{f}(u,0)$ is continuous for $u \neq 0.5$, $u \neq 1$, and $u \neq 1.5$. To prove the continuity of $\widetilde{f}(u,d)$ at $d = 0$, it suffices to prove its continuity at $u = 0.5$, $u = 1$ and $u = 1.5$.

Let us do this for $u = 0.5$. It follows from (7) that $\widetilde{f}(0.5,0) = -0.5$.

Consider an arbitrary $\varepsilon > 0$. Let

$$\varepsilon_1 = \min(\varepsilon, 0.25). \tag{A.1}$$

Choose $\delta = (\varepsilon_1)^2/2$. Let

$$(u-0.5)^2 + d^2 \leq \delta^2. \tag{A.2}$$

If $u < 0.5 - d$, it follows from (7) that $\widetilde{f}(u,d) = -u$. Then, in view of (A.2), we obtain

$$(\widetilde{f}(u,d) - \widetilde{f}(0.5,0))^2 = (-u+0.5)^2 \leq \delta^2 - d^2 \leq (\varepsilon_1)^4/4 - d^2 < (\varepsilon_1)^4.$$

By definition (see (A.1)), $\varepsilon_1 < 1$; therefore, $\varepsilon_1^4 < \varepsilon_1^2$ and

$$|(\widetilde{f}(u,d) - \widetilde{f}(0.5,0))| < \varepsilon_1 < \varepsilon.$$

Thus,

$$|(\widetilde{f}(u,d)-\widetilde{f}(0.5,0))|<\varepsilon \text{if } u<0.5-d. \tag{A.3}$$

If $0.5+d\le u\le 0.5-d$, then, according to (7),

$$\widetilde{f}(u,d)=0.5(-1+d+(u-0.5)^2d^{-1}).$$

Since $|u-0.5|\le d$ and $d\le\delta$ by virtue of (A.2), we find that

$$(\widetilde{f}(u,d)-\widetilde{f}(-0.5,0))^2=0.5(d+(u-0.5)^2d^{-1})=0.5(d^2+(u-0.5)^2)d^{-1}\le d\le\delta<\varepsilon_1^2\le\varepsilon^2.$$

Therefore,

$$|\widetilde{f}(u,d)-\widetilde{f}(0.5,0)|<\varepsilon \text{if } 0.5+d\le u\le 0.5-d. \tag{A.4}$$

If $0.5+d<u<1-d$, then $\widetilde{f}(u,d)=u-1$. Therefore,

$$(\widetilde{f}(u,d)-\widetilde{f}(0.5,0))^2=(u-0.5)^2\le\delta^2-d^2\le(\varepsilon_1)^4/4-d^2<(\varepsilon_1)^4.$$

We obtain

$$|\widetilde{f}(u,d)-\widetilde{f}(0.5,0)|<\varepsilon \quad \text{if } 0.5+d<u<1-d. \tag{A.5}$$

The continuity of $\widetilde{f}(u,d)$ at $d=0, u=0.5$ follows from (A.3)–(A.5). The proof for $u=1$ and $u=1.5$ is similar. The lemma is proved. □

Lemma 2.2. *On the set* $\overline{\mathbb{R}}$, $|\widetilde{f}(u_2,d)-\widetilde{f}(u_1,d)|\le|u_2-u_1|$.

Proof. It follows from (7) that

$$\frac{\partial\widetilde{f}(u,d)}{\partial u}=\begin{cases}-1 & \text{if } u<0.5-d,\ d\in(0,0.25)\\ (u-0.5)d^{-1} & \text{if } 0.5-d\le u\le 0.5+d,\ d\in(0,0.25)\\ \text{undefined} & \text{if } u=0.5,\ d=0\\ 1 & \text{if } 0.5+d<u<1-d,\ d\in(0,0.25)\\ -(u-1)d^{-1} & \text{if } 1-d\le u\le 1+d,\ d\in(0,0.25)\\ \text{undefined} & \text{if } u=1,\ d=0\\ -1 & \text{if } 1+d<u<1.5-d,\ d\in(0,0.25)\\ (u-1.5)d^{-1} & \text{if } 1.5-d\le u\le 1.5+d,\ d\in(0,0.25)\\ \text{undefined} & \text{if } u=1.5,\ d=0\\ 1 & \text{if } 1.5+d<u,\ d\in(0,0.25)\end{cases} \tag{A.6}$$

According to (A.6), for $d\in(0,\,0.25)$, the derivative $\partial\widetilde{f}(u,d)/\partial u$ is continuous, bounded, and satisfies the condition

$$\left|\frac{\partial\widetilde{f}(u,d)}{\partial u}\right|\le 1$$

Then, for $d\in(0,\,0.25)$, the function $\widetilde{f}(u,d)$ satisfies the Lipschitz condition with respect to u:

$$|\widetilde{f}(u_2,d)-\widetilde{f}(u_1,d)|\le K_1|u_2-u_1|$$

where

$$K_1=\sup_{u\in(-\infty,\infty)}\left|\frac{\partial\widetilde{f}(u,d)}{\partial u}\right|=1$$

Now consider the case $d=0$.

Let $U_1=0.5$, $U_2=1$, and $U_3=1.5$. Consider intervals S_n, $n=1,\dots,4$, where $S_1=\{-\infty<u<U_1\}$, $S_2=\{U_1<u<U_2\}$, $S_3=\{U_2<u<U_3\}$, and $S_4=\{U_3<u<\infty\}$.

It follows from (A.6) that the derivative $\partial\widetilde{f}(u,0)/\partial u$ exists for $u\in S_1\cup S_2\cup S_3\cup S_4$ and $|\partial\widetilde{f}(u,0)/\partial u|=1$.

Let $u_1\le u_2$, $u_1\in S_k$, and $u_2\in S_n$.

If $k=n$, then $\partial\widetilde{f}(u,0)/\partial u$ exists and is continuous and constant for $u\in[u1,\,u2]$. Then

$$|\widetilde{f}(u_2,0)-\widetilde{f}(u_1,0)|=|u_2-u_1| \tag{A.7}$$

If $n=k+1$, then

$$\begin{aligned}|\widetilde{f}(u_2,0)-\widetilde{f}(u_1,0)|&=|\widetilde{f}(u_2,0)-\widetilde{f}(U_k,0)+\widetilde{f}(U_k,0)-\widetilde{f}(u_1,0)|,\\ &\le|\widetilde{f}(u_2,0)-\widetilde{f}(U_k,0)|+|\widetilde{f}(U_k,0)-\widetilde{f}(u_1,0)|\\ &\le|u_2-U_k|+|U_k-u_1|=|u_2-u_1|\end{aligned} \tag{A.8}$$

If $n > k+1$, then

$$|\widetilde{f}(u_2,0)-\widetilde{f}(u_1,0)| = |\widetilde{f}(u_2,0)-\widetilde{f}(U_{n-1},0)+\sum_{j=k+1}^{n-1}(\widetilde{f}(U_j,0)-\widetilde{f}(U_{j-1},0))+\widetilde{f}(U_k,0)-\widetilde{f}(u_1,0)|$$
$$\le |\widetilde{f}(u_2,0)-\widetilde{f}(U_{n-1},0)|+\sum_{j=k+1}^{n-1}|\widetilde{f}(U_j,0)-\widetilde{f}(U_{j-1},0)|+|\widetilde{f}(U_k,0)-\widetilde{f}(u_1,0)|$$
$$\le |u_2-U_{n-1}|+\sum_{j=k+1}^{n-1}|U_j-U_{j-1}|+|U_k-u_1| = |u_2-u_1| \tag{A.9}$$

Relations (A.7)–(A.9) imply that $|\widetilde{f}(u_2,d)-\widetilde{f}(u_1,d)| \le |u_2-u_1|$ for $d=0$.

The lemma is proved. □

Lemma 2.3. *The solution $\widetilde{u}(\xi,d)$ of problem (8) in the case of $\widetilde{f}(u,d)$ given by (7) converges, as $d\to 0$, to the solution of problem (4) with $f(u)$ given by (6), i.e., (9) holds.*

Proof. For $\widetilde{u} < 0.25$, the function $\widetilde{f}(\widetilde{u},d) = -\widetilde{u}$ is independent of d (see (7)). Therefore, the solution of Eq. (8), namely,

$$\widetilde{u}(\xi) = 0.25e^{(\frac{1-\sqrt{1+4\gamma^2(W+1)}}{2\gamma})(\xi-\xi_a)}\ \forall d \text{ if } \xi \ge \xi_a \tag{A.10}$$

is also independent of d. This solution satisfies the condition $\lim_{\xi\to\infty}\widetilde{u}(\xi,d)=0$. Here, ξ_a is an arbitrary constant.

Fix some ξ_a and consider the family of integral curves $\widetilde{u}(\xi,d)$ described by (A.10). For $\xi \ge \xi_a$ these curves coincide and $\forall d\ \widetilde{u}(\xi_a,d)=0.25$. Therefore, all integral curves of the family are solutions of the Cauchy problem

$$\frac{d^2\widehat{u}}{d\xi^2}-\frac{1}{\gamma}\frac{d\widehat{u}}{d\xi} = W\widehat{u}-\widetilde{f}(\widehat{u},d),$$
$$\widehat{u}(\xi_a,d)=0.25,\ \frac{d\widehat{u}}{d\xi}\,|_{\xi=\xi_a}=0.25\left(\frac{1-\sqrt{1+4\gamma^2(W+1)}}{2\gamma}\right). \tag{A.11}$$

Any solution of the boundary value problem (8) solves Cauchy problem (A.11). However, it will be shown below that some solutions of problem (A.11) violate the condition $\lim_{\xi\to-\infty}u(\xi,d)=u_l$.

A problem equivalent to (A.11) can be written by using a normal system of differential equations:

$$\frac{d\mathbf{y}}{d\xi} = \mathbf{Y}(\mathbf{y},d),$$

where

$$y_1=\widehat{u},\ y_2=\frac{d\widehat{u}}{d\xi},$$

$$Y_1(y_1,y_2,d)=y_2,$$

$$Y_2(y_1,y_2,d)=\frac{1}{\gamma}y_2+Wy_1-\widetilde{f}(y_1,d).$$

Since $Y_1(y_1, y_2, d)$ is a linear function of y_2, it is continuous and satisfies the Lipschitz condition with respect to y_1 and y_2. $Y_2(y_1, y_2, d)$ is the sum of $\widetilde{f}(y_1,d)$ and linear functions of y_1 and y_2. By Lemmas 2.1 and 2.2, $\widetilde{f}(y_1,d)$ is a continuous function of y_1 and d that satisfies the Lipschitz condition with respect to y_1. Therefore, $Y_2(y_1, y_2, d)$ is a continuous function satisfying the Lipschitz condition with respect to y_1 and y_2. Then $\mathbf{Y}(\mathbf{y}, d)$ is continuous and satisfies the Lipschitz condition with respect to $\mathbf{y}$.

By the theorem on continuous dependence on a parameter,

$$\lim_{d\to 0}\widehat{u}(\xi,d)=\widehat{u}(\xi,0).$$

Since the set of functions $\widetilde{u}(\xi,d)$ solving problem (8) is a subset of solutions $\widehat{u}(\xi,d)$ of Cauchy problem (A.11), we conclude that

$$\lim_{d\to 0}\widetilde{u}(\xi,d)=\widetilde{u}(\xi,0).$$

The lemma is proved. □

References

[1] El GA, Hoefer MA, Shearer M. Dispersive and diffusive-dispersive shock waves for nonconvex conservation laws. SIAM Rev 2016;59:3–61.
[2] Hayes B, Shearer M. Undercompressive shocks and riemann problems for scalar conservation laws with nonconvex fluxes. Proc R Soc A 1999;129:733–54.
[3] Jacobs D, McKinney B, Shearer M. Travelling wave solutions of the modified korteweg-de vries-burgers equation. J Differ Equ 1995;116:448–67.
[4] LeFloch PG, Shearer M. Nonclassical riemann solvers with nucleation. Proc R Soc A 2004;134:961–84.
[5] Hayes B, Shearer M. A nonconvex scalar conservation law with a trilinear flux. Q Appl Math 2010;59:615–35.
[6] Kulikovskii AG. The possible effect of oscillations in a discontinuity structure on the set of admissible discontinuities. Soviet Phys Dokl 1984;29:28328.
[7] Kulikovskii AG, Chugainova AP. Simulation of the influence of small-scale dispersion processes in a continuum on the formation of large-scale phenomena. Comput Math Math Phys 2004;44:1062–8.
[8] Kulikovskii AG, Chugainova AP. Classical and non-classical discontinuities in solutions of equations of non-linear elasticity theory. Russ Math Surv 2008;63:283–350.
[9] Il'ichev AT, Chugainova AP, Shargatov VA. Spectral stability of special discontinuities. Doklady Math 2015;91:347–51.
[10] Chugainova AP, Shargatov VA. Stability of discontinuity structures described by a generalized kdv-burgers equation. Comput Math Math Phys 2016;56:263–77.
[11] Il'ichev AT, Chugainova AP. Spectral stability theory of heteroclinic solutions to the korteweg-de vries-burgers equation with an arbitrary potential. Proc Steklov Inst Math 2016;295:148–57.
[12] Kulikovskii AG, Chugainova AP, Shargatov VA. Uniqueness of self-similar solutions to the riemann problem for the hopf equation with complex nonlinearity. Comput Math Math Phys 2016;56:1355–62.
[13] LeFloch PG. Hyperbolic systems of conservation laws: the theory of classical and nonclassical shock waves. Lectures in Mathematics. Birkhauser Verlag Basel: ETH Zurich; 2002.
[14] Chugainova AP, Il'ichev AT, Kulikovskii AG, Shargatov VA. Problem of arbitrary discontinuity disintegration for the generalized hopf equation: selection conditions for a unique solution. J Appl Math 2017;82:496–525.
[15] Kulikovskii AG. Strong discontinuities in flows of continua and their structure. Proc Steklov Inst Math 1990;182:285–317.
[16] Kulikovskii AG, Pogorelov NV, Semenov AYu. Mathematical aspects of numerical solution of hyperbolic systems. Monographs and surveys in pure and applied mathematics 188. Boca: Chapman & Hall/CRC; 2001.
[17] Lax PD. Hyperbolic systems of conservation laws. Comm Pure Appl Math 1957;10:537–66.
[18] Godunov SK. No unique blurring of discontinuities in solutions to quasilinear systems. Sov Math Dokl 1961;2:43–4.
[19] Kulikovskii AG, Chugainova AP. On the steady-state structure of shock waves in elastic media and dielectrics. J Exp Theor Phys 2010;110:851–62.
[20] Chugainova AP. Nonstationary solutions of a generalized korteweg-de vries-burgers equation. Proc Steklov Inst Math 2013;281:204–12.
[21] Chugainova AP, Shargatov VA. Stability of nonstationary solutions of the generalized kdv-burgers equation. Comput Math Math Phys 2015;55:251–63.
[22] Chugainova AP. Special discontinuities in nonlinearly elastic media. Comput Math Math Phys 2017;57:1013–21.
[23] Lomonosov IV, Tahir NA. Theoretical investigation of shock wave stability in metals. Appl Phys Lett 2008;92:101905.
[24] Lomonosov IV. Multi-phase equation of state for aluminum. Laser Part Beams 2007;25:567–84.
[25] Kulikovskii AG. Surfaces of discontinuity separating two perfect media of different properties: recombination waves in magnetohydrodynamics. J Appl Math Mech 1968;32:1145–51.