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On some properties of the coupled Fitzhugh-Nagumo equations

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Abstract. We consider the FitzHugh-Nagumo model describing two neurons electrically coupled via ion flow through gap junctions between them. This model is a simple example of a neural network, which has a vast amount of periodic behaviors. It is shown that system of equations describing this model does not pass the Painlevé test. Analysis of stability of system's trivial stationary point is carried out. It is shown that this equilibrium point is not always stable. For some parameter regions where solution oscillates bifurcation diagrams are plotted and Lyapunov exponents are calculated. It is shown that analyzed non-stationary solutions are quasiperiodic.

1. Introduction

FitzHugh-Nagumo model is a simple model for the description of impulse propagation from one neuron to another [1]. It is described by the following system of ordinary differential equations

$$\begin{aligned}\frac{dv}{dt} &= v(v-a)(1-v) - R, \\ \frac{dR}{dt} &= \epsilon(v - \beta R),\end{aligned}$$

where a, β, ϵ, R are positive parameters, v is neuron's membrane potential, R is its recovery variable. This model has been studied a lot in periodical literature. Some recent results can be found in [2–4].

In this work we study a system of two FitzHugh-Nagumo neurons coupled electrically via ion flow through gap junctions between them. This model is described by the following system of equations [5]

$$\begin{aligned}\frac{dv_1}{dt} &= v_1(v_1 - a)(1 - v_1) - R_1 + g(v_1 - v_2), \\ \frac{dR_1}{dt} &= \epsilon(v_1 - \beta R_1), \\ \frac{dv_2}{dt} &= v_2(v_2 - a)(1 - v_2) - R_2 + g(v_2 - v_1), \\ \frac{dR_2}{dt} &= \epsilon(v_2 - \beta R_2),\end{aligned}\tag{1}$$



where v_1, v_2 are membrane potentials of considered neurons, R_1, R_2 are their recovery variables. The neurons are paired through the voltage equations [5].

It is important to study periodic sequences of neural impulses to control dynamic functions of the body. Understanding the mechanism of neural networks which have a wide variety of periodic activities is highly significant [6]. FitzHugh-Nagumo model is one of the simplest examples of such network.

2. Painlevé analysis of the equations studied

System of eqs. (1) be reduced to a system of two second-order equations

$$\begin{aligned} v_{1tt} + (3v_1^2 - 2v_1(a+1) + \beta\epsilon + a - g)v_{1t} + \epsilon\beta v_1^3 - \epsilon\beta(a+1)v_1^2 + \\ + \epsilon(1 + \beta a - \beta g)v_1 + g(v_{2t} + \epsilon\beta v_2) = 0, \\ v_{2tt} + (3v_2^2 - 2v_2(a+1) + \beta\epsilon + a - g)v_{2t} + \epsilon\beta v_2^3 - \epsilon\beta(a+1)v_2^2 + \\ + \epsilon(1 + \beta a - \beta g)v_2 + g(v_{1t} + \epsilon\beta v_1) = 0. \end{aligned} \quad (2)$$

Substituting $v_1 = \sqrt{u_1}$, $v_2 = \sqrt{u_2}$ into the system of equations (2) we get

$$\begin{aligned} u_1 u_{1tt} - \frac{1}{2} u_{1t}^2 + (3u_1^2 - 2(a+1)u_1^{\frac{3}{2}} + (\epsilon\beta + a - g)u_1)u_{1t} + 2\epsilon\beta u_1^3 - \\ - 2\epsilon\beta(a+1)u_1^{\frac{5}{2}} + 2\epsilon(\beta a - \beta g + 1)u_1^2 + 2u_1^{\frac{3}{2}}g(\epsilon\beta\sqrt{u_2} + \frac{u_{2t}}{\sqrt{u_2}}) = 0, \\ u_2 u_{2tt} - \frac{1}{2} u_{2t}^2 + (3u_2^2 - 2(a+1)u_2^{\frac{3}{2}} + (\epsilon\beta + a - g)u_2)u_{2t} + 2\epsilon\beta u_2^3 - \\ - 2\epsilon\beta(a+1)u_2^{\frac{5}{2}} + 2\epsilon(\beta a - \beta g + 1)u_2^2 + 2u_2^{\frac{3}{2}}g(\epsilon\beta\sqrt{u_1} + \frac{u_{1t}}{\sqrt{u_1}}) = 0. \end{aligned} \quad (3)$$

We begin by applying the Painlevé test to eqs. (3) (see e.g. [7]). Leading terms of this system are

$$\begin{aligned} u_1 u_{1tt} + 3u_1^2 u_{1t} - \frac{1}{2} u_{1t}^2 = 0, \\ u_2 u_{2tt} + 3u_2^2 u_{2t} - \frac{1}{2} u_{2t}^2 = 0. \end{aligned} \quad (4)$$

Substituting $u_1 = a_0 t^{-n}$, $u_2 = b_0 t^{-m}$ into eqs. (4), we find first coefficients of the solution's expansion into Laurent series $(a_0, n) = (b_0, m) = (1/2, 1)$.

To find Fuchs indices we substitute

$$\begin{aligned} u_1 &= \frac{1}{2t} + \beta_1 t^{r-1}, \\ u_2 &= \frac{1}{2t} + \beta_2 t^{r-1} \end{aligned}$$

into (4). Equating to zero terms linear in β_1 and β_2 we get

$$\begin{pmatrix} \frac{1}{2}r^2 - \frac{1}{4}r - \frac{3}{4} & 0 \\ 0 & \frac{1}{2}r^2 - \frac{1}{4}r - \frac{3}{4} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Accordingly, the Fuchs indices are $r = -1, \frac{3}{2}, -1, \frac{3}{2}$. We can see that system of equations (3) does not pass the Painlevé test, since it does not have three non-negative integer Fuchs indices. None of the local solutions of (3) is general [7].

However, if we look for the expansions of the solution of eqs. (2) in the Puiseux series

$$\begin{aligned} v_1(t) &= \frac{a_0}{\sqrt{t-t_0}} + a_1 + a_2\sqrt{t-t_0} + a_3(t-t_0) + a_4(t-t_0)^{\frac{3}{2}} + a_5(t-t_0)^2 + \dots, \\ v_2(t) &= \frac{b_0}{\sqrt{t-t_0}} + b_1 + b_2\sqrt{t-t_0} + b_3(t-t_0) + b_4(t-t_0)^{\frac{3}{2}} + b_5(t-t_0)^2 + \dots \end{aligned} \quad (5)$$

We get the following values of expansion's coefficients

$$\begin{aligned} a_0 = b_0 &= \pm \frac{1}{\sqrt{2}}, \quad a_1 = b_1 = \frac{a}{3} + \frac{1}{3}, \quad a_2 = b_2 = \frac{\sqrt{2}}{12}(a^2 - a + 1), \\ a_4 = b_4 &= \frac{1}{432}(a^4 - 2a^3 + 3a^2 - 2a + 144\epsilon + 1). \end{aligned} \quad (6)$$

There are arbitrary constants (t_0, a_3) in the first expansion and (t_0, b_3) in the second. They correspond to Fuchs indices and are arbitrary. So, we have obtained expansion of the solution of system of eq.(2) into the Puiseux series.

3. Stability analysis of the equations studied

Stationary points of (1) can be found from

$$\begin{aligned} v_1(1-v_1)(v_1-a) - R_1 + g(v_1-v_2) &= 0, \\ \epsilon(v_1 - \beta R_1) &= 0, \\ v_2(1-v_2)(v_2-a) - R_2 + g(v_2-v_1) &= 0, \\ \epsilon(v_2 - \beta R_2) &= 0. \end{aligned} \quad (7)$$

This system of equations can have up to nine real solutions. We denote them as $(v_{1i}, v_{2i}, u_{1i}, u_{2i})$, $i = \overline{1, 9}$. Clearly, $(v_1, R_1, v_2, R_2) = (0, 0, 0, 0)$ is a stationary point. We denote it as $(0, 0, 0, 0) = O$.

Linearization of eqs. (1) about the steady states is

$$\begin{aligned} \frac{d\xi_1}{dt} &= (-3v_{1i}^2 + 2(a+1)v_{1i} - a + g)\xi_1 - \eta_1 - g\xi_2, \\ \frac{d\eta_1}{dt} &= \epsilon(\xi_1 - \beta\eta_1), \\ \frac{d\xi_2}{dt} &= (-3v_{2i}^2 + 2(a+1)v_{2i} - a + g)\xi_2 - \eta_2 - g\xi_1, \\ \frac{d\eta_2}{dt} &= \epsilon(\xi_2 - \beta\eta_2). \end{aligned} \quad (8)$$

Jacobian matrix of the stationary state is

$$\begin{pmatrix} -3v_{1i}^2 + 2(a+1)v_{1i} - a + g & -1 & -g & 0 \\ \epsilon & -\epsilon\beta & 0 & 0 \\ -g & 0 & -3v_{2i}^2 + 2(a+1)v_{2i} - a + g & -1 \\ 0 & 0 & \epsilon & -\epsilon\beta \end{pmatrix}. \quad (9)$$

We will examine stability of the state O. Jacobian matrix (9) at this point is

$$\begin{pmatrix} -a + g & -1 & -g & 0 \\ \epsilon & -\epsilon\beta & 0 & 0 \\ -g & 0 & -a + g & -1 \\ 0 & 0 & \epsilon & -\epsilon\beta \end{pmatrix}, \quad (10)$$

and the corresponding characteristic equation is

$$\lambda^4 + 2(\beta\epsilon + 2a - 2g)\lambda^3 + (\beta^2\epsilon^2 + 4a\beta\epsilon - 4\beta\epsilon g + a^2 - 2ag + \epsilon)\lambda^2 + (2a\beta^2\epsilon^2 - 2\beta^2\epsilon^2 g + 2a^2\beta\epsilon - 4a\beta\epsilon g + \beta\epsilon^2 + a\epsilon - \epsilon g)\lambda + a^2\beta^2\epsilon^2 - 2a\beta^2\epsilon^2 g + a\beta\epsilon^2 - \beta\epsilon^2 - \beta\epsilon^2 g = 0. \quad (11)$$

Eigenvalues of (10) are

$$\lambda_{1,2} = \frac{1}{2}(-\beta\epsilon - a \pm \sqrt{(\beta\epsilon - a)^2 - 4\epsilon}), \quad (12)$$

$$\lambda_{3,4} = \frac{1}{2}(-\beta\epsilon - a + 2g \pm \sqrt{(\beta\epsilon - a + 2g)^2 - 4\epsilon}).$$

For all admissible parameter values $Re\lambda_{1,2} < 0$, because ϵ , β and a are positive parameters. Stationary point O loses stability when $g = 1/2(\beta\epsilon + a)$, since $Re\lambda_{3,4} > 0$ when $-\beta\epsilon - a + 2g > 0$. Let us fix $a = 0.3$, $\beta = 0.1$, $\epsilon = 0.01$.

In case of phase-repulsive coupling ($g > 0$) and fixed parameter values trivial stationary point is unstable when $g > \frac{1}{2}(a + \beta\epsilon) = 0.1505$. It is an unstable node if $(\beta\epsilon - a + 2g)^2 - 4\epsilon > 0$ and an unstable focus otherwise. For example, when $g = 0.2$ eigenvalues of (10): $\lambda_1 = -0.039$, $\lambda_2 = -0.2616$, $\lambda_{3,4} = 0.049 \pm 0.086i$. This rest state is an unstable focus. When $g = 0.5$ eigenvalues of (10): $\lambda_1 = -0.039$, $\lambda_2 = -0.2616$, $\lambda_3 = 0.6854$, $\lambda_4 = 0.014$. Stationary point is an unstable node. Dependences of neurons' membrane potentials on time when $g > 0$ are presented on fig.3 (a),(b).

In case of excitable coupling ($g < 0$) neuron potentials quickly synchronize after excitement. When $g = -0.5$ eigenvalues of (10) are $\lambda_1 = -0.2616$, $\lambda_2 = -0.039$, $\lambda_3 = -1.2923$, $\lambda_4 = -0.009$. This rest state is a stable node. Dependence of neurons' membrane potentials on time when $g < 0$ is depicted on fig. 3 (c).

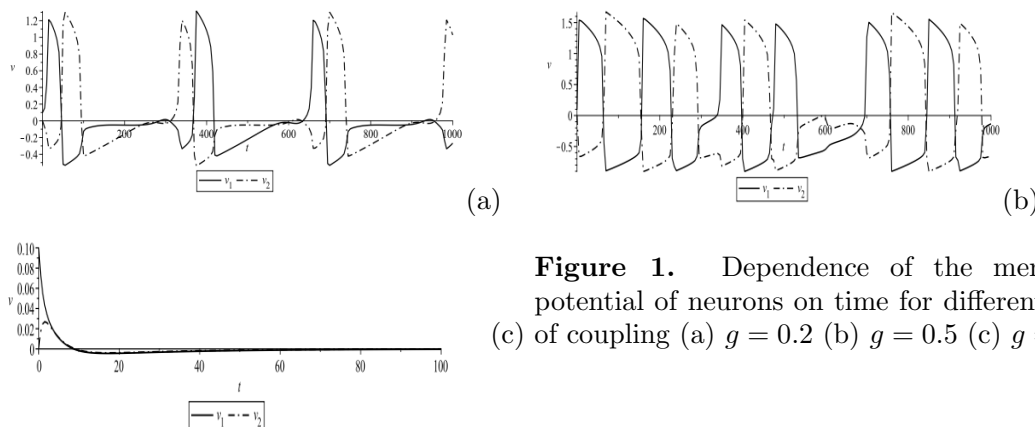


Figure 1. Dependence of the membrane potential of neurons on time for different types (c) of coupling (a) $g = 0.2$ (b) $g = 0.5$ (c) $g = -0.5$

To determine whether solutions for $g > 0$ have periodic or chaotic nature we plot bifurcation diagrams and find Lyapunov exponents of a system. To compute maximal Lyapunov exponents we use the standard algorithm by Bennetin [8].

Sectioning the solution by a plane $R_2 = const$ in a positive direction gives one, two or three points in the Poincare section for all g except for $g \in (0.5, 0.54)$ (fig.2). Fig.3 shows that for all g the maximal Lyapunov exponent is close to zero. For studied values of parameter g computations show that the second largest Lyapunov exponent is close to zero as well. For example for $g = 0.507$ Lyapunov spectrum is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-6 \cdot 10^{-5}, -9 \cdot 10^{-4}, -1.09, -4.16)$. Accordingly, all investigated oscillating solutions for $g > 0$ are quasiperiodic.

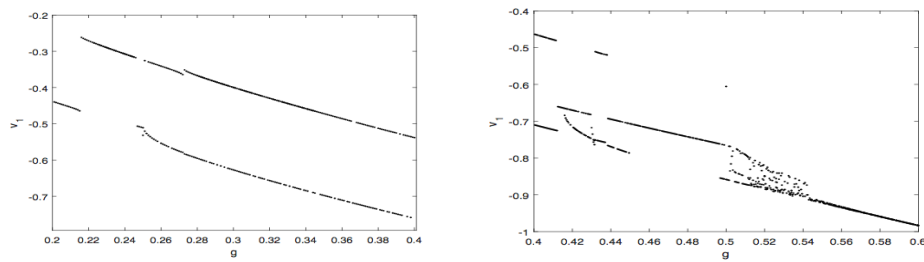


Figure 2. Bifurcation diagrams when $a = 0.3, \beta = 0.1, \epsilon = 0.01$; section by the plane $R_2 = 0.3$ on the left, section by the plane $R_2 = 0.5$ on the right

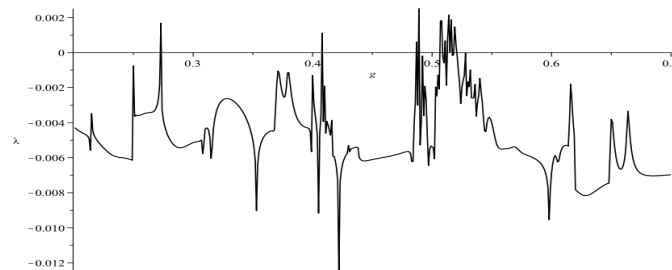


Figure 3. Dependence of the maximal Lyapunov exponent on coupling strength

4. Conclusion

FitzNagumo model describing impulse propagation between two coupled neurons is considered in this work.

For studied system of equations Painlevé test is performed. We obtain that we cannot find an expansion of a general solution of the investigated system in a Laurent or Puasieux series. Stability of system's equilibrium points is investigated. It is shown that all of the examined non-stationary solutions are quasiperiodic.

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