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Conservation laws of the complex Ginzburg-Landau equation

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ABSTRACT

The complex Ginzburg-Landau equation is considered. It is shown that there are three conservation laws corresponding to this equation. These conservation laws are found using the direct transformation of the equation. The first integral of the ordinary differential equation is given by reduction to the traveling wave variables. The bright optical soliton of the Ginzburg-Landau equation is found. Conservative quantities corresponding to the power, momentum and energy of optical soliton are calculated.

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1. Introduction

In this paper we study the complex Ginzburg-Landau equation in the form

$$i\,q_t + a\,q_{xx} + F(|q|^2)\,q = \frac{1}{|q|^2\,q^*}\left[\alpha\,|q|^2\left(|q|^2\right)_{xx} - \beta\left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma\,q, \tag{1}$$

where $q(x,t)$ is a complex valued function, x and t are spatial and temporal coordinates, $F(|q|^2)$ is a function of $|q|^2$ and a, α, β, γ are parameters of nonlinear differential equation (1).

Eq. (1) is used to describe a number of physical processes [1–8]. This equation has been considered many times in the scientific literature. The main attention in the study of Eq. (1) was paid to the construction of exact solutions to the equation taking into account the various approaches (see, for example papers [9–20]).

However it is quite important to know, whether the nonlinear evolution equation admits conservation laws. They reflect the symmetry of mathematical models and correspond to the physical conservative quantities [1–3].

It is known that Eq. (1) has conservation laws if it can be written as follows [1–3]

$$\frac{\partial T_j}{\partial t} + \frac{\partial X_j}{\partial x} = 0, \qquad (j = 1, 2, \ldots), \tag{2}$$

where T_j are densities and X_j are fluxes. The expressions for T_j and X_j depend on $q(x,t)$ and its derivatives and are independent of each other.

The conservative quantities follow from Eq. (2) in the form [1–3]

$$I_j = \int_{-\infty}^{\infty} T_j dx = Const. \tag{3}$$

Conservation laws corresponding to the Ginzburg-Landau equation with some constraints on the parameters of Eq. (1) have been considered in papers [21–25].

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The objective of this paper is to find conservation laws of Eq. (1) taking into account direct calculations and to obtain conservative quantities of the bright optical soliton.

This paper is organized as follows. In Sections 2, 3 and 4 we find the conservation laws corresponding to Eq. (1) at $\alpha = 2\beta$. The bright optical soliton of Eq. (1) is presented in Section 5. In Section 6 the conservative quantities corresponding to the power, momentum and energy are given.

2. The first conservation law corresponding to Eq. (1)

To find conservation laws of Eq. (1) we use direct calculations. With this aim we write Eq. (1) as the following system of equations

$$i\,q_t + a\,q_{xx} + F(|q|^2)\,q - \frac{\alpha\,q}{|q|^2}\left(|q|^2\right)_{xx} + \frac{\beta\,q}{|q|^4}\left\{\left(|q|^2\right)_x\right\}^2 - \gamma\,q = 0 \tag{4}$$

and

$$-i\,q_t^* + a\,q_{xx}^* + F(|q|^2)\,q^* - \frac{\alpha\,q^*}{|q|^2}\left(|q|^2\right)_{xx} + \frac{\beta\,q^*}{|q|^4}\left\{\left(|q|^2\right)_x\right\}^2 - \gamma\,q^* = 0. \tag{5}$$

Multiplying Eq. (4) by q^* and Eq. (5) by $-q$ and adding the resulting expressions, we obtain the equation

$$i\,(q^*\,q_t + q\,q_t^*) + a\,(q^*q_{xx} - q\,q_{xx}^*) = 0. \tag{6}$$

Eq. (6) can be presented in the form

$$\frac{\partial T_1}{\partial t} + \frac{\partial X_1}{\partial x} = 0, \tag{7}$$

where T_1 and X_1 are written as

$$T_1 = i\,|q|^2, \qquad X_1 = a\,(q^*q_x - q\,q_x^*). \tag{8}$$

Eq. (7) yields the conservative quantity in the form

$$P = \int\limits_{-\infty}^{\infty} |q|^2\,dx \tag{9}$$

which corresponds to the impulse power.

3. The second conservation law corresponding to Eq. (1)

The second conservation law corresponding to Eq. (1) can be found by multiplying Eq. (4) by q_x^* and Eq. (5) by q_x. Adding these equations, we get

$$\begin{gathered} i\,(q_x^*\,q_t - q_x\,q_t^*) + a\,(q_x^*\,q_{xx} + q_x\,q_{xx}^*) + F(|q|^2)\,(q_x^*q + q^*q_x) - \\ \gamma\,(q_x^*q + q^*q_x) + \left[\frac{\beta}{|q|^4}\left\{\left(|q|^2\right)_x\right\}^2 - \frac{\alpha}{|q|^2}\left(|q|^2\right)_{xx}\right](q_x^*q + q^*q_x) = 0. \end{gathered} \tag{10}$$

Taking into account the following formulas

$$q_x^*\,q_t - q_x\,q_t^* = \frac{1}{2}\,\frac{\partial}{\partial t}\,(q_x^*q - q^*q_x) - \frac{1}{2}\,\frac{\partial}{\partial x}\,(q_t^*q - q^*q_t), \tag{11}$$

$$q_x^*q + q^*q_x = \frac{\partial}{\partial x}\,(|q|^2), \tag{12}$$

$$q_x^*q_{xx} + q_{xx}^*q_x = \frac{\partial}{\partial x}\,(|q_x|^2), \tag{13}$$

we can write Eq. (10) in the form

$$\begin{gathered} \frac{i}{2}\,\frac{\partial}{\partial t}\,(q_x^*q - q^*q_x) - \frac{i}{2}\,\frac{\partial}{\partial x}\,(q_t^*q - q^*q_t) + a\,\frac{\partial}{\partial x}\,(|q_x|^2) + \frac{\partial}{\partial x}\,Q\,(|q|^2) - \\ -\gamma\,\frac{\partial}{\partial x}(|q|^2) + \left[\frac{\beta}{|q|^4}\left\{\left(|q|^2\right)_x\right\}^2 - \frac{\alpha}{|q|^2}\left(|q|^2\right)_{xx}\right]\frac{\partial}{\partial x}(|q|^2) = 0, \end{gathered} \tag{14}$$

where the function $Q\,(u)$ is expressed by the following integral

$$Q\,(u) = \int F(u)\,du. \tag{15}$$

One can note that there is a function $F(|q|^2)$ in the conservation laws of Eq. (1) which needs to be chosen. For example, taking $F_1(|q|^2) = |q|^{2n}$, we obtain

$$Q_1(|q|^2) = \frac{1}{n+1}\,|q|^{2n+2}. \tag{16}$$

In the case of $F_2(|q|^2) = \lambda_1\,|q|^{2n} + \lambda_2\,|q|^{2p} + \lambda_3\,|q|^{2l}$, where $n,\ p,\ l \in \mathbb{N}$, we obtain the formula

$$Q_2(|q|^2) = \frac{\lambda_1}{n+1}\,|q|^{2n+2} + \frac{\lambda_2}{p+1}\,|q|^{2p+2} + \frac{\lambda_3}{l+1}\,|q|^{2l+2}, \tag{17}$$

where λ_1, λ_2 and λ_3 are constants.

Taking into account the formula

$$\frac{\partial}{\partial x}\left(\frac{u_x^2}{u}\right) = \frac{2\,u_x\,u_{xx}}{u} - \frac{u_x^3}{u^2} \tag{18}$$

we rewrite Eq. (14) at $\alpha = 2\,\beta$ as follows

$$\frac{i}{2}\,\frac{\partial}{\partial t}\,(q_x^* q - q^* q_x) - \frac{i}{2}\,\frac{\partial}{\partial x}\,(q_t^* q - q^* q_t) + a\,\frac{\partial}{\partial x}\,(|q_x|^2) + \frac{\partial}{\partial x}\,Q\,(|q|^2) -$$

$$-\gamma\,\frac{\partial}{\partial x}(|q|^2) - \beta\,\frac{\partial}{\partial x}\left[\frac{\left\{\left(|q|^2\right)_x\right\}^2}{|q|^2}\right] = 0. \tag{19}$$

The last equation can be presented as the conservation law

$$\frac{\partial T_2}{\partial t} + \frac{\partial X_2}{\partial x} = 0, \tag{20}$$

where T_2 and X_2 are determined by the formulas

$$T_2 = \frac{i}{2}\,(q_x^* q - q^* q_x), \tag{21}$$

$$X_2 = \frac{i}{2}\,(q^* q_t - q_t^* q) + a\,|q_x|^2 + Q\,(|q|^2) - \gamma\,|q|^2 - 4\,\beta\,(|q|_x)^2. \tag{22}$$

From Eq. (20) we obtain the conservative quantity in the form

$$M = \frac{i}{2}\int\limits_{-\infty}^{\infty}\left(q_x^* q - q^* q_x\right) = Const, \tag{23}$$

which corresponds to the conservated momentum of the solution $q(x,t)$.

4. The third conservation law corresponding to Eq. (1)

In Section 2 we have obtained the following equation

$$q_{xx}^* q - q_{xx}\,q^* = \frac{i}{a}\,\frac{\partial}{\partial t}\left(|q|^2\right). \tag{24}$$

At the next step we multiply Eq. (4) by q_{xx}^* and Eq. (5) by $-q_{xx}$. Adding the resulting expressions yields

$$i\,(q_{xx}^* q_t + q_{xx}\,q_t^*) + F(|q|^2)\,(q_{xx}^* q - q_{xx}\,q^*) - \gamma\,(q_{xx}^* q - q_{xx}\,q^*) +$$

$$-\frac{\alpha}{|q|^2}\left(|q|^2\right)_{xx}(q_{xx}^* q - q_{xx}\,q^*) + \frac{\beta}{|q|^4}\left\{\left(|q|^2\right)_x\right\}^2(q_{xx}^* q - q_{xx}\,q^*) = 0. \tag{25}$$

Using the formula

$$q_{xx}^* q_t + q_{xx}\,q_t^* = \frac{\partial}{\partial x}\,(q_x^*\,q_t + q_x\,q_t^*) - \frac{\partial}{\partial t}(|q_x|^2) \tag{26}$$

and Eq. (25), we obtain the equation in the form

$$i\,\frac{\partial}{\partial x}\,(q_x^*\,q_t + q_x\,q_t^*) - i\,\frac{\partial}{\partial t}(|q_x|^2) + \frac{i}{a}\,F(|q|^2)\,\frac{\partial}{\partial t}\left(|q|^2\right) - \frac{i\,\gamma}{a}\,\frac{\partial}{\partial t}\left(|q|^2\right) +$$

$$\left[\frac{i\,\beta}{a\,|q|^4}\left\{\left(|q|^2\right)_x\right\}^2 - \frac{i\,\alpha}{a\,|q|^2}\left(|q|^2\right)_{xx}\right]\frac{\partial}{\partial t}\left(|q|^2\right) = 0. \tag{27}$$

Taking into account the formula

$$\frac{u_x^2 u_t}{u^2} - \frac{2u_{xx}u_t}{u} = \frac{\partial}{\partial t}\left(\frac{\{u_x\}^2}{u}\right) - 2\frac{\partial}{\partial x}\left(\frac{u_x u_t}{u}\right), \tag{28}$$

where $u(x,t) = |q|^2$, one can write Eq. (27) at $\alpha = 2\beta$ in the form

$$a\frac{\partial}{\partial x}(q_x^* q_t + q_x q_t^*) - a\frac{\partial}{\partial t}(|q_x|^2) + F(|q|^2)\frac{\partial}{\partial t}\left(|q|^2\right) - \gamma\frac{\partial}{\partial t}\left(|q|^2\right) + \beta\frac{\partial}{\partial t}\left[\frac{\left\{(|q|^2)_x\right\}^2}{|q|^2}\right] - 2\beta\frac{\partial}{\partial x}\left[\frac{(|q|^2)_x(|q|^2)_t}{|q|^2}\right] = 0. \tag{29}$$

Eq. (29) can be written as the conservation law

$$\frac{\partial T_3}{\partial t} + \frac{\partial X_3}{\partial x} = 0, \tag{30}$$

where T_3, $Q(|q|^2)$ and X_3 take the form

$$T_3 = Q(|q|^2) - a|q_x|^2 - \gamma|q|^2 + 4\beta(|q|_x)^2, \quad Q(u) = \int F(u)\,du \tag{31}$$

and

$$X_3 = a(q_x^* q_t + q_x q_t^*) + 8\beta(|q|_t|q|_x). \tag{32}$$

Taking into account Eq. (31) we obtain the conservative quantity in the form

$$I_3 = \int_{-\infty}^{\infty}\left(Q(|q|^2) - a|q_x|^2 - \gamma|q|^2 + 4\beta(|q|_x)^2\right)dx, \tag{33}$$

where I_3 corresponds to the energy of the wave.

5. Optical solitons of Eq. (1)

We look for the solution of Eq. (1) taking into account the traveling wave reduction

$$q(x,t) = y(z)\,e^{i\psi(z)}, \qquad z = x - C_0 t. \tag{34}$$

Substituting (34) into Eq. (1), we obtain the real and imaginary parts as follows

$$y\,a\,\psi_{zz} + 2a\,\psi_z\,y_z - C_0\,y_z = 0 \tag{35}$$

and

$$(a - 2\alpha)\,y_{zz} + y\,\psi_z - a\,y\,\psi_z^2 - \gamma\,y - 2(\alpha - 2\beta)\frac{y_z^2}{y} + b\,y\,F(y^2) = 0. \tag{36}$$

We have the first integral of Eq. (35) in the form

$$a\,y^2\,\psi_z - C_0\,y^2 + C_1 = 0, \tag{37}$$

where C_1 is a constant of integration.

From Eq. (37) we find

$$\psi_z = \frac{C_0}{2a} - \frac{C_1}{a\,y^2}. \tag{38}$$

Substituting ψ_z into Eq. (36), we get the following

$$(a - 2\alpha)\,y\,y_{zz} + (4\beta - 2\alpha)\,y_z^2 - \frac{{C_1}^2}{a\,y^2} + b\,y^2\,F(y^2) + \left(\frac{C_0^2}{4a} - \gamma\right)y^2 = 0. \tag{39}$$

Let us demonstrate that Eq. (39) has the first integral. This result can be obtained taking into account the useful formula

$$r\,y_z^2 + s\,y\,y_{zz} = \frac{s^2}{s+r}\,y^{\frac{s-r}{s}}\frac{d^2}{dz^2}\left(y^{\frac{s+r}{s}}\right), \tag{40}$$

where r and s are constants.

Using formula (40) and the notations

$$r=4\beta-2\alpha,\qquad s=a-2\alpha,\qquad m=1+\frac{r}{s}=\frac{a+4\beta-4\alpha}{a-2\alpha}, \tag{41}$$

we write Eq. (39) as follows

$$\frac{(a-2\alpha)}{m}\frac{d^2}{dz^2}\left(y^m\right)-\left(\gamma-\frac{C_0^2}{4a}\right)y^m+b\,y^m\,F(y^2)-\frac{C_1^2}{a}\,y^{m-4}=0. \tag{42}$$

The first integral of Eq. (42) takes the form

$$(a-2\alpha)\,m\,y^{2m-2}\,y_z^2-\left(\gamma-\frac{C_0^2}{4a}\right)y^{2m}+$$
$$2\,b\,m\int y^{2m-1}\,F(y^2)\,dy-\frac{m\,C_1^2}{m-2}\,y^{2m-4}=C_2, \tag{43}$$

where C_2 is an arbitrary constant.

From Eq. (43) we obtain the nonlinear ordinary differential equation of the first order in the form

$$(a-2\alpha)\,m\,y_z^2-\left(\gamma-\frac{C_0^2}{4a}\right)y^2-\frac{m\,C_1^2}{(m-2)\,y^2}-C_2\,y^{2-2m}+$$
$$2\,b\,m\,y^{2-2m}\int y^{2m-1}\,F(y^2)\,dy=0. \tag{44}$$

We have obtained integral (44) at the arbitrary parameter values α and β except for $m\neq 0$ and $m\neq 2$ and $a\neq 2\alpha$. To the best of our knowledge the first integral (44) of Eq. (39) is new which allows us to look for a number of solutions of the Ginzburg-Landau equation.

Without loss of generality let us consider the particular case of Eq. (1) at $F(|q|^2)=|q|^{2n}$, where $n\in\mathbb{N}$. In this case Eq. (44) can be written in the form

$$(a-4\beta)my_z^2-\left(\gamma+\frac{C_0^2}{4a}\right)y^2+\frac{bm}{n+m}y^{2m+2n}-\frac{m\,C_1^2}{(m-2)y^2}=C_2y^{2-2m}. \tag{45}$$

Assuming $C_1=0$, $C_2=0$ and $\alpha=2\beta$, we have $m=1$ and Eq. (45) is reduced to the following equation

$$y_z^2=\mu\,y^2-\nu\,y^{2+2n}, \tag{46}$$

where μ and ν are determined by the formulas

$$\mu=\frac{4a\gamma+C_0^2}{4a(a-4\beta)},\qquad \nu=\frac{b}{(n+1)(a-4\beta)}. \tag{47}$$

Solution of Eq. (46) takes the form [26–38]

$$y(z)=\left[\frac{4\,\mu\,\mathrm{e}^{\sqrt{\mu}(z-z_0)}}{4\,\mu\,\nu+\mathrm{e}^{2\sqrt{\mu}(z-z_0)}}\right]^{\frac{1}{n}}. \tag{48}$$

Substituting solution (48) into (34) produces the function $\psi(z)$ in the form $\psi(z)=\frac{C_0}{2a}(z-x_0)$. Meanwhile the solution of Eq. (1) takes the form

$$q(x,t)=\left[\frac{4\,\mu\,\mathrm{e}^{n\sqrt{\mu}(z-z_0)}}{4\,\mu\,\nu+\mathrm{e}^{2n\sqrt{\mu}(z-z_0)}}\right]^{\frac{1}{n}}\exp\left\{\frac{i\,C_0}{2\,a}\,(x-C_0\,t-x_0)\right\}, \tag{49}$$

where x_0 is an arbitrary constant. Solution (49) can be used to calculate the conservative quantities of optical solitons described by Eq. (1).

6. Conservative quantities corresponding to optical solitons

The power of the optical soliton (48) is given by the formula

$$P=\int\limits_{-\infty}^{\infty}|q|^2dx=\int\limits_{-\infty}^{\infty}\left[\frac{4\,\mu\,\mathrm{e}^{n\sqrt{\mu}(z-z_0)}}{4\,\mu\,\nu+\mathrm{e}^{2n\sqrt{\mu}(z-z_0)}}\right]^{\frac{2}{n}}dz. \tag{50}$$

Using the new variable $z=\frac{1}{n\sqrt{\mu}}\ln(\xi)$, we calculate integral (50)

$$P=\frac{(4\mu)^{\frac{1}{n}}}{2n\sqrt{\mu}\,\nu^{\frac{1}{n}}}\int\limits_0^\infty\left[\frac{\xi}{(1+\xi^2)}\right]^{\frac{2}{n}}\frac{d\xi}{\xi}=\frac{(4\mu)^{\frac{1}{n}}}{2n\sqrt{\mu}\,\nu^{\frac{1}{n}}}\,\frac{\Gamma(\frac{1}{n})\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}, \tag{51}$$

where $\Gamma(\nu)$ is the gamma function.

Taking into account formula (23) for momentum of optical soliton (48), we get

$$M=\frac{i}{2}\int\limits_{-\infty}^{\infty}(q_x^*,q-q^*q_x)\,dx=\frac{C_0\,(4\mu)^{\frac{1}{n}}}{4an\sqrt{\mu}\,\nu^{\frac{1}{n}}}\,\frac{\Gamma(\frac{1}{n})\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}. \tag{52}$$

In order to calculate the energy of optical soliton (48), we have to use formula (33). Assuming $F(|q|^2)=|q|^{2n}$, we get the following integral

$$H=\int\limits_{-\infty}^{\infty}\left(\frac{b\,|q|^{2n+2}}{2n+2}+a\,|q_x|^2-\gamma\,|q|^2-4\,\beta\,(|q|_x)^2\right)dx=$$

$$\frac{b\,(4\,\mu)^{\frac{1+n}{n}}}{4n(n+1)\sqrt{\mu}\,\nu^{\frac{1+n}{n}}}\,\frac{\Gamma\left(1+\frac{1}{n}\right)\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(2+\frac{2}{n}\right)}+\frac{(\mu a-\gamma-4\mu\beta)(4\mu)^{\frac{1}{n}}}{2n\sqrt{\mu}\,\nu^{\frac{1}{n}}} \tag{53}$$

$$\frac{\Gamma(\frac{1}{n})\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}\right)}+\frac{\nu\,(a-4\,\beta)\,(4\,\mu)^{\frac{2}{n}}}{2n\sqrt{\mu}\,\nu^{\frac{2}{n}}}\,\frac{\Gamma(\frac{2}{n})\,\Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{4}{n}\right)}.$$

In the case of Kerr nonlinearity, the formulas are simplified and take the form

$$P_1=\frac{\sqrt{\mu}}{\nu},\qquad M_1=\frac{C_0\sqrt{\mu}}{2a\,\nu},$$
$$H_1=\frac{10\sqrt{\mu}}{3\,\nu^2}\left[\frac{b\,\mu}{40}-\frac{3\,\gamma\,\nu}{5}+\mu\,\nu\,(a-4\,\beta)\right]. \tag{54}$$

We believe that these conservative quantities can be used to check numerical modeling of physical processes described by the Ginzburg-Landau equation.

7. Conclusion

In this paper we have considered the complex Ginzburg-Landau equation. The Cauchy problem for this equation cannot be solved by the inverse scattering transform. However we have shown that there are three conservation laws corresponding to this equation which have been found by direct calculations. Exact solitons of the Ginzburg-Landau equation have been obtained taking into account the traveling wave reduction. The bright optical solitons of the equation have been presented. These solitons allow us to obtain the conservative quantities corresponding to the power, momentum and energy of optical soliton.

CRediT authorship contribution statement

Nikolay A. Kudryashov: Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

I confirm that I have not any conflicts of interests on the manuscript "*Conservation laws of the complex Ginzburg-Landau equation*" and I confirm that we have not got any plagiarism and other loans.

Data availability

No data was used for the research described in the article.

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References

[1] Alwyn Scott, Nonlinear Science. Emergence and Dynamics of Coherent Structures, Oxford University Press, 2003.
[2] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New-York, 1985.
[3] E. Kengne, W.-M. Liu, L.Q. English, B.A. Malomed, Ginzburg–Landau models of nonlinear electric transmission networks, Phys. Rep. 982 (2022) 1–124.
[4] B.A. Malomed, New findings for the old problem: exact solutions for domain walls in coupled real Ginzburg-Landau equations, Phys. Lett. A: Gen. At. Solid State Phys. 422 (2022) 127802.
[5] Y. Qiu, B.A. Malomed, D. Mihalache, X. Zhu, L. Zhang, Y. He, Soliton dynamics in a fractional complex Ginzburg-Landau model, Chaos Solitons Fractals 131 (2020) 109471.
[6] B.A. Malomed, Past and present trends in the development of the pattern-formation theory: domain walls and quasicrystals, Physics 3 (4) (2021) 1015–1045.
[7] M.-Y. Tang, M.-Y. Wang, The chirped wave propagation in modified cubic–quintic complex Ginzburg–Landau equation with parabolic law, Optik 275 (2023) 170547.
[8] K. Hosseini, M. Mirzazadeh, D. Baleanu, N. Raza, C. Park, A. Ahmadian, S. Salahshour, The generalized complex Ginzburg–Landau model and its dark and bright soliton solutions, Eur. Phys. J. Plus 136 (7) (2021) 709.
[9] N.A. Kudryashov, Exact solutions of the complex Ginzburg–Landau equation with law of four powers of nonlinearity, Optik 265 (2022) 169548.
[10] J.-J. Fang, D.-S. Mou, Y.-Y. Wang, H.-C. Zhang, C.-Q. Dai, Y.-X. Chen, Soliton dynamics based on exact solutions of conformable fractional discrete complex cubic Ginzburg–Landau equation, Results Phys. 20 (2021) 103710.
[11] W. Zhu, Y. Xia, Y. Bai, Traveling wave solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity, Appl. Math. Comput. 382 (2020) 125342.
[12] P.-H. Lu, B.-H. Wang, C.-Q. Dai, Fractional traveling wave solutions of the (2 + 1)-dimensional fractional complex Ginzburg–Landau equation via two methods, Math. Methods Appl. Sci. 43 (15) (2020) 8518–8526.
[13] M.-Y. Wang, Optical solitons with perturbed complex Ginzburg–Landau equation in Kerr and cubic–quintic–septic nonlinearity, Results Phys. 33 (2022) 105077.
[14] Y.A.S. Wellot, G.D. Nkaya, Analytical solution of the Ginzburg-Landau equation, Eur. J. Pure Appl. Math. 15 (4) (2022) 1750–1759.
[15] W. Zhu, Z. Ling, Y. Xia, M. Gao, Bifurcations and the exact solutions of the time-space fractional complex Ginzburg-Landau equation with parabolic law nonlinearity, Fractal Fract. 7 (2) (2023) 201.
[16] Y. Liu, S. Chen, L. Wei, B. Guan, Exact solutions to complex Ginzburg–Landau equation, Pramana J. Phys. 91 (2) (2018) 29.
[17] A.H. Arnous, A. Biswas, A.H. Kara, D. Milovic, Y. Yildirim, H.M. Alshehri, Sequel to "cubic-quartic optical soliton perturbation with complex Ginzburg–Landau equation by the enhanced Kudryashov's method", IET Optoelectron. 16 (4) (2022) 149–159.
[18] N.A. Kudryashov, First integrals and general solution of the complex Ginzburg-Landau equation, Appl. Math. Comput. 386 (2020) 125407.
[19] M. Ekici, Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion and Kudryashov's refractive index structures, Phys. Lett. A: Gen. At. Solid State Phys. 440 (2022) 128146.
[20] A.M. Yalci, M. Ekici, Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion, Opt. Quantum Electron. 54 (3) (2022) 167.
[21] A. Biswas, A.H. Kara, S. Khan, Y. Yidirim, M.F. Mahmood, H.M. Alshehri, M.R. Belic, Conservation laws for cubic–quartic optical solitons with complex Ginzburg–Landau equation having five nonlinear refractive index structures, Optoelectron. Adv. Mater., Rapid Commun. 16 (3–4) (2022) 137–141.
[22] A. Biswas, A.H. Kara, Y. Sun, Q. Zhou, Y. Yildirim, H.M. Alshehri, M.R. Belic, Conservation laws for pure-cubic optical solitons with complex Ginzburg–Landau equation having several refractive index structures, Results Phys. 31 (2021) 104901.
[23] L. Fu, H. Dong, C.M. Khalique, H. Yang, Conservation laws, exact solutions of time-space fractional generalized Ginzburg-Landau equation for shallow wake flows, J. Appl. Anal. Comput. 11 (2) (2021) 874–891.
[24] A. Biswas, Chirp-free bright optical solitons and conservation laws for complex Ginzburg-Landau equation with three nonlinear forms, Optik 174 (2018) 207–215.
[25] J.-P. Dias, F. Oliveira, H. Tavares, On a coupled system of a Ginzburg-Landau equation with a quasilinear conservation law, Commun. Contemp. Math. 22 (7) (2020) 1950054.
[26] N.A. Kudryashov, A generalized model for description of propagation pulses in optical fiber, Optik 189 (2019) 42–52.
[27] N.A. Kudryashov, Mathematical model of propagation pulse in optical fiber with power nonlinearities, Optik 212 (2020) 164750.
[28] N.A. Kudryashov, Method for finding highly dispersive optical solitons of nonlinear differential equations, Optik 206 (2020) 163550.
[29] N.A. Kudryashov, Method for finding optical solitons of generalized nonlinear Schrodinger equations, Optik 261 (2022) 169163.
[30] N.A. Kudryashov, Highly dispersive solitary wave solutions of perturbed nonlinear Schrodinger equations, Appl. Math. Comput. 371 (2020) 124972.
[31] M. Ozisik, A. Secer, M. Bayram, (3+1)-dimensional Sasa–Satsuma equation under the effect of group velocity dispersion, self-frequency shift and self-steepening, Optik 275 (2023) 170609.
[32] M. Ozisik, A. Secer, M. Bayram, M. Cinar, N. Ozdemir, H. Esen, I. Onder, Investigation of optical soliton solutions of higher-order nonlinear Schrodinger equation having Kudryashov nonlinear refractive index, Optik 274 (2023) 170548.
[33] I. Onder, A. Secer, M. Bayram, Optical soliton solutions of time-fractional coupled nonlinear Schrodinger system via Kudryashov-based methods, Optik 272 (2023) 170362.
[34] M. Cinar, A. Secer, M. Ozisik, M. Bayram, Optical soliton solutions of (1 + 1)- and (2 + 1)-dimensional generalized Sasa-Satsuma equations using new Kudryashov method, Int. J. Geom. Methods Mod. Phys. 20 (2) (2023) 2350034.
[35] N.K. Vitanov, Simple Equations Method (SEsM): an effective algorithm for obtaining exact solutions of nonlinear differential equations, Entropy 24 (11) (2022) 1653.
[36] N.A. Kudryashov, The generalized Duffing oscillator, Commun. Nonlinear Sci. Numer. Simul. 93 (2021) 105526.
[37] M. Ekici, Stationary optical solitons with Kudryashov's quintuple power law nonlinearity by extended Jacobi's elliptic function expansion, J. Nonlinear Opt. Phys. Mater. 32 (1) (2023) 2350008.
[38] A. Biswas, A. Sonmezoglu, M. Ekici, Sequel to "stationary optical solitons with Kudryashov's laws of refractive index" (generalized temporal evolution), J. Nonlinear Opt. Phys. Mater. 32 (1) (2023) 2350005.