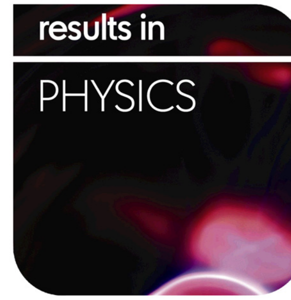
Journal Pre-proofs



Optical solitons and stability analysis with coupled nonlinear schrodinger's equations having double external potentials

Hamdy I. Abdel-Gawad, A. Biswas, A.S. Alshomrani, M.R. Belic

PII: S2211-3797(19)32334-4
DOI: https://doi.org/10.1016/j.rinp.2019.102707
Reference: RINP 102707

To appear in: *Results in Physics*

Received Date: 31 July 2019
Revised Date: 8 September 2019
Accepted Date: 25 September 2019

Please cite this article as: Abdel-Gawad, H.I., Biswas, A., Alshomrani, A.S., Belic, M.R., Optical solitons and stability analysis with coupled nonlinear schrodinger's equations having double external potentials, *Results in Physics* (2019), doi: https://doi.org/10.1016/j.rinp.2019.102707

OPTICAL SOLITONS AND STABILITY ANALYSIS WITH COUPLED NONLINEAR SCHRODINGER'S EQUATIONS HAVING DOUBLE EXTERNAL POTENTIALS

Hamdy I. Abdel-Gawad[(a)], A. Biswas[(b1−4)], A. S. Alshomrani[(b2)], and M. R. Belic[(c)]

[(a)]Mathematics Department, Faculty of Sciences, Cairo university, Egypt

[(b1)] Department of Physics, Chemistry and Mathematics, Alabama A&M University, Normal, AL 35762-7500, USA

[(b2)]Department of Mathematics, King Abdulaziz University, Jeddah-21589, Saudi Arabia

[(b3)]Department of Applied Mathematics, National Research Nuclear University, 31 Kashirskoe Shosse, Moscow-115409, Russian Federation

[(b4)] Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria-0008, South Africa

[(c)]Science Program, Texas A&M university at Qatar, 23874 Doha, Qatar

Abstract

We consider coupled nonlinear Schrodinger equation (CNLSE) of the Gross-Pitaevskii-type, with linear mixing and nonlinear cross-phase modulation. Motivated by the study of matter waves in Bose-Einstein condensates and multicomponent (vectorial) nonlinear optical systems, we investigate the eigenvalue problem of the CNLSE with double external potentials in a self-defocusig Kerr medium. For this system, we obtain different kinds of wave structures induced by two injected beams, of physical relevance in nonlinear optics and Bose-Einstein condensation. Exact solutions are found by the extended unified method. The linear stability of these solutions is analyzed through the formulation of an eigenvalue problem. The spectral problem is constructed by perturbing the frequency of stationary solutions and by linearizing the resulting equations near the stationary (or steady) states. Our study may simulate experimental work on multiple injected laser beams in a medium with Kerr-type nonlinearity.

Keywords. Coupled NLS equation, double external potentials, the eigenvalue problem, stability.

1 Introduction

Recently, the coupled nonlinear Schrodinger (CNLS) equations have become a topic of intense research, owing to their great applicative potential in many fields of physics, such as wave mixing in optics, rogue wave phenomena, and Bose-Einstein condensates (BECs) (see [1–11]). In BECs, the formation of localized solutions, *i.e.* solitons, occurs in the coupled time-dependent mean-field Gross–Pitaevskii (GP) equations [12]. Because of (at least) two components, such solutions of the one-dimensional CNLS equations were often found as vector (line) solitons [12] in nonlinear fiber optics. Other examples include waveguides coupled through the evanescent field overlap, the coupling of two polarizations modes in uniform guides, mode coupling in optical fibers, and so on [13]. The study of the propagation of optical solitons in multi-mode nonlinear couplets, which is important from the theoretical point of view, is also important in view of their possible applications [14,15]. Recent advances have indicated that solitons are ideal states for performing all-optical switching operations in nonlinear couplers. In fact, their stability leads to the possibility of controlling the coupling of the whole pulse, by means of changing the input power of a single beam.

Vector solitons considered so far in nonlinear optics were mostly bright solitons created in self-focusing media. The possibility of creation of dark vector solitons in self-defocusing media has been advanced more recently, and mostly theoretically. The experimental study of bright solitons in quasi-one-dimensional attractive systems is quite delicate, due to the possibility of collapse in such systems, although true one-dimensional systems do not exhibit collapse [9]. The two-component repulsive BECs with interspecies attraction are better suited for studying solitons, as such systems may not easily collapse [16] and one can have a controlled study of solitons.

Counter propagation of scalar waves in media with Kerr type nonlinearity that obey CNLS equations, is determined by the wavelength-scale changes in the refractive index [17-19]. Envelopes of incoherent copropagating waves in Kerr media may also obey CNLS equations. Motivated by the works of Baronio [26] and Guo [27], the bright–dark rogue solutions [20,21] and other higher-order localized waves [22] are also found in the two-component CNLS equation. Some semi-rational, multi-parametric localized wave solutions are obtained in the coupled Hirota equation [23–25].

Laser beams-induced periodic surface structures SS are a global phenomenon and can be generated on with the availability of laser pulses. Their structures can be generated in a simple single-step process, which allows a surface for adaptation, optical or mechanical waves. Their formation mechanisms are analyzed via scattering and diffraction constrained double-pulses. It has been shown that these solutions are organized via the process of absorption of energy deposition mechanisms Relevant featuring surface structures applications are of interest in the fields of optics and fluids.

Periodic wave or surface structures (SS) induced by laser beams are by now universal phenomena, commonly generated with the availability of femtosecond laser pulses. Such structures can be generated in a simple single-step process, which allows a surface to adapt to optical or mechanical waves. Their formation mechanisms are analyzed via scattering and diffraction of constrained double-pulses. It has been shown that these solutions are organized via the process of energy absorption or deposition mechanisms. Relevant applications featuring surface wave structures are of interest in the fields of optics and fluid mechanics, among others. The visualization of the solutions induced by the two injected laser beams is one of the objectives of this work.

2 The mathematical model

In this paper, we consider coupled equations of the GP type, with double external potentials $W(x)$ and $\alpha(x)$, of the form:

$$\begin{array}{l} i\psi_{1t} = -\frac{1}{2}\psi_{1xx} + \sigma|\psi_1^2 \mid \psi_1 + \beta|\psi_2^2|\psi_1 - \alpha(x)\psi_2 + W(x)\psi_1, \\ i\psi_{2t} = -\frac{1}{2}\psi_{2xx} + \sigma|\psi_2|^2\psi_2 + \beta|\psi_1|^2\psi_2 - \alpha(x)\psi_1 + W(x)\psi_2 \end{array} . \quad (1)$$

Equations (1) arise in different physical contexts, most notably in various nonlinear optical models, where they describe interacting coupled optical beams. which are coupled by linear mixing and by nonlinear cross-phase modulation.The functions u_1 and u_2 describe the light pulses within each waveguide [27] or fields in the case of BECs..The coefficients σ and β account for the self cross phase modulation nonlinearities.In the context of BECs, $W(x)$ is an external potential and $\alpha(x)$ is an in homogeneous rate of the linear interconversion between the two atomic states, which may be considered as the second potential. In (1) $\alpha(x)$ and $W(x)$ are two potential wells or walls (or both) as it will be shown, here, that they are dependent. The parameters σ , β and μ (cf.(3)) are free. Other physical systems governed by this type of model equations where they have straight forward applications to two-component BECs., to bimodal light propagation in nonlinear optics, in super fluids and in thermal conviction (see [28-31,35]). We mention that when we replace $\frac{\partial^2}{\partial x^2}$ by

$\nabla^2$the results to the solutions obtained are very abundant and consequently the physical phenomena are as well. In this context (1) was extended to two-dimensional case [36]. Also the stability has been established to the Domain walls patterns and moreover, the symmetry-breaking bifurcation was also identified.

Here we search for the stationary solutions in the form:

$$\psi_j(x,t) = e^{-i\mu t}\varphi_i(x), \quad j = 1, 2, \tag{2}$$

with μ being the propagation constant. Then, equation (1) becomes

$$\begin{aligned} \mu\,\varphi_1 + \tfrac{1}{2}\varphi_1'' - \sigma\varphi_1^3 - \beta\,\varphi_2^2\varphi_1 + \alpha(x)\,\varphi_2 - W(x)\,\varphi_{1,} = 0 \\ \mu\,\varphi_2 + \tfrac{1}{2}\varphi_2'' - \sigma\varphi_2^3 - \beta\,\varphi_1^2\varphi_2 + \alpha(x)\,\varphi_1 - W(x)\,\varphi_2 = 0. \end{aligned} \tag{3}$$

Here, we presume that the medium is self-defocusing ($\sigma > 0$). In (3), μ represents the chemical potential in BECs or in thermal conviction..The solutions are obtained, here, by utilizing the extended unified method [32-34]. Many recent interesting works in a relevant area to the present work were done in [35-37].

This paper is organized as follows. In section 3, some exact solutions of Eq. (3) are found. Section 4 is devoted to solving the eigenvalue problem for analyzing the stability of the solutions obtained. Applications of the stability analysis are presented in section 5. Section 6 is devoted to conclusions.

3 Exact solutions

The solutions obtained here are polynomial and rational in an auxiliary function that satisfies an auxiliary equation. We first consider the polynomial solutions. They may take the form

$$\varphi_1(x) = \sum_{i=0}^{n} a_i(x)\,g^i(x), \quad \varphi_2(x) = \sum_{i=0}^{m} b_i(x)\,g^i(x), \tag{4}$$

$$g'(x) = q(x) \sum_{i=0}^{pk} c_i \, g^i(x), \; p = 1, 2,$$

where $g(x)$ is an auxiliary function, $a_i(x)$, $b_j(x)$, and $q(x)$, $i = 0, 1, .., n$, $j = 1, ..., m$, are unknown functions, and c_i, $i = 0, 1, 2$ are arbitrary parameters.

An analysis by the extended unified method leads to qualities $n = m = k - 1$ and the values of k are determined by the consistency condition, $1 \leq k \leq 5$ (see [32-34]). When $p = 1$ the solutions of the auxiliary equation yield elementary (or implicit) functions, while when $p = 2$, they may be special or elementary functions. It is worthy to mention that when $k = 1$ or $n = m = 0$, the polynomial solutions reduce to the rational. In this case, the present method reduces to the exp-function method. But here, one can take $k \geq 2$ as well. That is, in the applications the rational solutions of integrable equations may be found when $p = 1, 2$ and $k \geq 2$. Thus, the present method generalizes the exp-function method.

Now, we consider the case $k = 2$ and $p = 1$, and have

$$\begin{gathered} \varphi_1(x) = \; a_1(x) \, g(x) + a_0(x), \; \varphi_2(x) = \; b_1(x) \, g(x) + b_0(x), \\ g'(x) = q(x) \, (c_2 g(x)^2 + c_1 g(x) + c_0). \end{gathered} \tag{5}$$

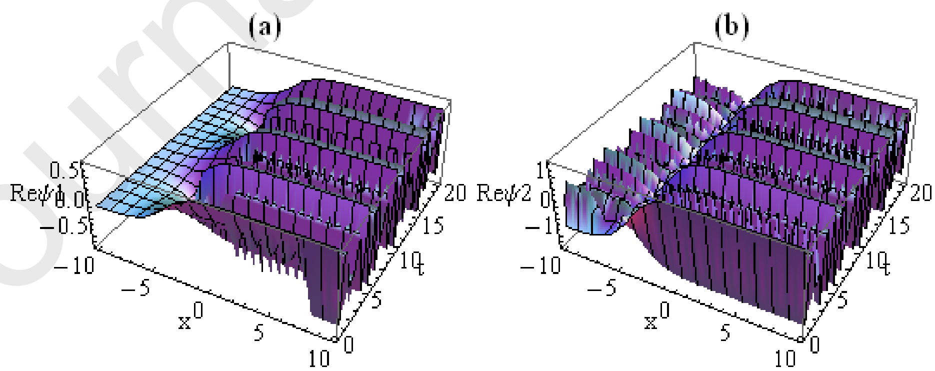
It should be noticed that the computations performed here are not straightforward. This is due to the fact that (i) The algebraic equations obtained are nonlinear, with no unique solution. (ii) We have to solve the compatibly conditions, which commonly arise in the calculations. As an example, when we have two equations, namely for $a_0''(x)$ and $a_0'(x)$, then the condition reads $a_0''(x) - (a_0'(x))' = 0$. Thus, the demanding computations in this paper are done using symbolic computational methods.

Now, by substituting (5) into (3), one obtains:

$$\beta = 3\sigma,\ a_1(x) = b_1(x) = \frac{c_2 q(x)}{\sqrt{\beta+\sigma}}, a_0(x) := \frac{32 e^{2rx} k r^3}{Q(x)},$$
$$b_0(x) = \frac{1}{Q(x)\sqrt{\sigma}}(A_1^2 e^{4rx} r - 4r^3(1 - 4c_1 e^{2rx} + 4e^{4rx} a^2 + 8e^{2rx} k\sqrt{\sigma})),$$

$$q(x) = -\frac{1}{S(x)}((32 e^{2rx} r^3),\ S(x) = (4A_1 e^{2rx} r - 4r^2 - e^{4rx}(A_1^2 - 16a^2 r^2)),$$
$$Q(x) = A_1^2 e^{4rx} - 4A_1 e^{2rx} r - 4(-1 + 4e^{4rx} a^2 r^2),$$
$$g(x) := \frac{1}{16 r^2 c_2}(-8r^2 c_1 + (A_1^2 e^{2rx} - 2A_1 r - 16a^2 e^{2rx} r^2)),$$
$$\alpha(x) = -\frac{F_1(x)}{F_0(x)},$$
$$F_1(x) = 512 e^{4rx} r^6 (-3A_1^2 e^{4rx} + 4r\hat{}2(3 - 4c1 e^{2rx} + 12 e^{4rx} a^2 + 16 e^{2rx} k\sqrt{\sigma}))\cdot$$
$$(-c_2 c_0 + 2c_1 k\sqrt{\sigma} - 4k^2\sigma),$$
$$F_0(x) = Q(x)^2 (A_1^2 e^{4rx} - 4r\hat{}2(1 - 4c1 e^{2rx} + 4e^{4rx} a^2 + 16 e^{2rx} k\sqrt{\sigma})),$$
$$a = \sqrt{c_1^2 - 4c_2 c_0},\ r = \sqrt{\mu - C_0} \tag{6}$$

and $W(x) = \alpha(x) + C_0$. In (6) k, A_1, C_0 and $c_i\,, i = 0, 1, 2$ are arbitrary parameters. We mention that the potentials $\alpha(x)$ and $W(x)$ are two walls when $\alpha(x) > 0, C_0 > 0$ or two wells when $\alpha(x) < 0, C_0 < 0$ or otherwise. By substituting (6) into (5), we get the solutions $\varphi_1(x)$ and $\varphi_2(x)$. The results are too lengthy to be reproduced here. In figures 1a and 1b, the solutions $Re\,\psi_1$ and $Re\,\psi_2$ are displayed against x and t.



Figures 1a and 1b. Plots of $Re\,\psi_i = \cos(\mu t)\varphi_i$, $i = 1, 2$, for the values of parameters A_1= -50., $r = 0.5, c_1$= -8, c_0 =0.9, c$_2$ = 1.3, σ=

$1.2, \beta = 3\sigma, k = -1.3$, and μ=5.

These figures show that the solutions are periodic solitons (or a vector of soliton waves) coupled to anti-pulses to the right of the x-axis in time, while they are localized in space. Which may be due to the fact that one potential acts as a repulsive potential wall while the other one's acts as an attractive potential well. This may also explain why there are no propagation of waves along the negative x-axis in Fig. 1a. However, in Fig. 1b waves with small intensity are propagating to the left. transversal wave concavity occurs on the negative x-axis as well.

Next, we are concerned with finding the rational solutions. To this end, we consider the following cases:

Case I. When $p = 1$, $k = 2$, we have:

$$\begin{gathered} \varphi_1(x) = \frac{a_1(x)\,g(x)+a_0(x)}{s_1(x)\,g(x)+s_0(x)},\ \varphi_2(x) = \frac{b_1(x)\,g(x)+b_0(x)}{s_1(x)\,g(x)+s_0(x)}, \\ g'(x) = q(x)\,(c_2 g(x)^2 + c_1 g(x) + c_0). \end{gathered} \tag{7}$$

By substituting (7) into (3), one gets:

$$\begin{gathered} a_1(x) = \frac{(2A_0\beta\, a_0(x)\sqrt{\beta+\sigma}+A_0(c1-2A_0c_0)\beta\, q(x)s_0(x))}{2\beta\sqrt{\beta+\sigma}},\ b_1(x) = \frac{2}{3}A_0 b_0(x), \\ a_0(x) := \frac{(3A_0(c1-2A_0c_0)^2\beta\,(\beta-3\sigma)q(x)s_0(x))}{(2A_0(c1-2A_0c_0)\beta\sqrt{\beta+\sigma}(-\beta+3\sigma)))}, \\ b_0(x) = \frac{(3(c1-2A_0c_0)q(x)\,s_0(x)}{2\sqrt{\beta+\sigma}}, \\ c_2 := \frac{A_0c_1}{2},\ c_1 := \frac{8A_0c_0/}{3},\ A_0 := \frac{(-\beta^2-5\beta\sigma)}{(\beta^2-\beta\backslash\sigma+2\sigma^2)}, \\ q(x) = \frac{-1}{C_0},\ s_1(x) = A_0 s_0(x), \\ W(x) = \mu + \frac{3}{4}(3c_1^2 - 14A_0c_1c_0 + 16A_0^2c_0^2)q(x)^2 - \alpha(x), \\ g(x) = \frac{(\beta^2-\beta\,\sigma+2\sigma^2)(-3+C_1 e^{-\frac{((4\beta(\beta+5\sigma)c_0C_0x)}{3(\beta^2-\beta\,\sigma+2\sigma^2)}})}{2\beta\backslash(\beta+5\sigma))(1+C_1 e^{-\frac{((4\beta(\beta+5\sigma)c_0C_0x)}{3(\beta^2-\beta\,\sigma+2\sigma^2)}})}, \end{gathered} \tag{8}$$

where C_1, C_0 and c_0 are arbitrary parameters and $\alpha(x)$ is a free function that may be taken to represent a solitary, double-hump, Gaussian or an oscillatory potential. We consider the case of a solitary potential

$$\alpha(x) = k_0^2 \,\mathrm{sech}\left(\frac{4\beta(\beta+5\sigma)c_0C_0x}{3(\beta^2-\beta\,\sigma+2\sigma^2)}\right)^2. \tag{9}$$

To find $W(x)$, we use the relevant equation (18), derived below.

By substituting from (8) and (9) into (7) one finds

$$\varphi_1(x) = \frac{4\beta\,(\beta+5\sigma)\,c_0C_0C_1e^{-\frac{((4\beta(\beta+5\sigma)c_0C_0x)}{3(\beta^2-\beta\,\sigma+2\sigma^2)}}}{3\sqrt{\beta+\sigma}(1+C_1e^{-\frac{((4\beta(\beta+5\sigma)c_0C_0x)}{3(\beta^2-\beta\,\sigma+2\sigma^2)}})(\beta^2-\beta\,\sigma+2\sigma^2)},\ \varphi_2(x) = -\varphi_1(x). \tag{10}$$

The solutions given by (10) are used to display $Re\psi_i, i = 1, 2$ against x and t in figures 2a and 2b respectively, for some values of the free parameters.

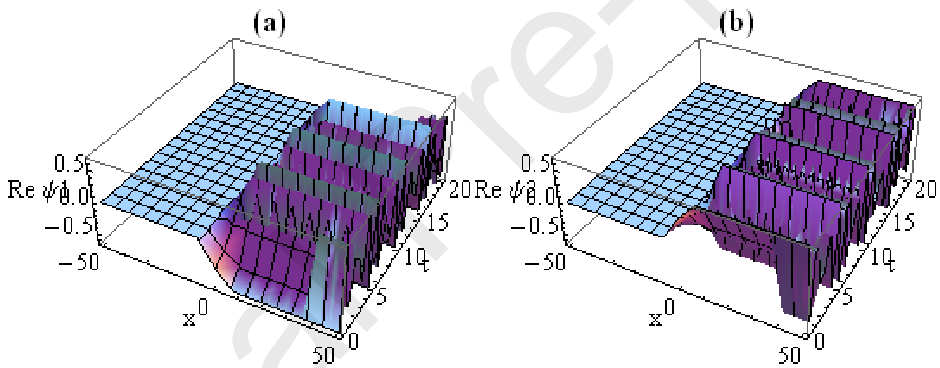


Figure 2a and 2b. 3D plots of $Re\psi_i$, $i = 1, 2$. The parameters are: $\mu = 5, C_0 = .5, C_1 = 1.5, c_0 = .6, \sigma = 1.5$, and $\beta = 3$.

The figure displays the propagation of waves which are periodic in time and localized in space. %The waves exhibit solitary waves with pulses. We bear in mind that the potentials are solitary waves.

Case II. When $p = 2$, $k = 2$, we have

$$\begin{aligned}\varphi_1(x) &= \tfrac{a_1(x)\,g(x)+a_0(x)}{s_1(x)\,g(x)+s_0(x)},\ \varphi_2(x) = \tfrac{b_1(x)\,g(x)+b_0(x)}{s_1(x)\,g(x)+s_0(x)},\\ g'(x) &= q(x)\,\sqrt{g(x)^2(c_2g(x)^2+c_1g(x)+c_0)}.\end{aligned} \tag{11}$$

By substituting (11) into (3), one finds

$$b_1(x) = \frac{A_0\sqrt{\alpha(x)}}{(A_1A_0-A_2e^{A_3x})},\ bo(x) = \frac{e^{-A_3x}\sqrt{\alpha(x)}}{A_2A_3},\ s_0(x) = \frac{A_4}{(A_1A_0-A_2e^{A_3x})},$$

$$a_1(x) := \frac{A_2^2A_3^23A_4^2e^{2A_3x}\sqrt{\alpha(x)}}{2(A1Ao-A2e^{A_3x})^3\beta}/,\ a_0(x) = \frac{A_2A_3A_4^2e^{A_3x}}{2(A1Ao-A2e^{A_3x})^2\beta},\ s_1(x) = A_0s_0(x),$$

$$\begin{gathered} q(x) = \frac{A_4/((A_1A_0-A_2e^{A_3x})^2}{\alpha(x)}, \\ g(x) = \frac{\sqrt{c_0}}{a}sech(A_5 - A_4\sqrt{c_0}\int(A_1A - A_2e^{A_3x})^{-2}\alpha(x)^{-1}dx), \end{gathered} \tag{12}$$

and the potential $W(x)$ is given by

$$\begin{gathered} W(x) = \frac{1}{Q(x)}(e^{-2A_3x}(8A_2^2A_3^2A_4^2e^{2A_3x}(A_1A_0 - A_2e^{A_3x})^2\beta^2\mu\,\alpha(x)^2 \\ +(4A_1^4A_0^4\beta^2 + e^{A_3x}(-16A_1^3A_0^3\beta^2 + A_2e^{A_3x}(24A_1^2A_0^2\beta^2 \\ +A2e^{A_3x}(-16A_1A_0\beta^2 + A_2e^{A_3x}(A_3^4A_4^4 + 4\beta^2)))))(\beta - \sigma)\alpha(x)^3 \\ -A_2^2A_3^2A_4^2e^{2A_3x}A_1A_0 - A_2e^{A_3x})^2\alpha'(x)^2 \\ +2A_2^2A_3^2A_4^2e^{2A_3x}(A_1A_0 - e^{A_3x})^2\beta^2\alpha''(x))), \\ Q(x) = (8A_2^2A_3^2A_4^2(A_1A_0 - A_2e^{A_3x}\beta^2\alpha(x)^2). \end{gathered} \tag{13}$$

We recall that $\alpha(x)$ is a free function. It may be taken as a single or double-hump soliton, or an oscillatory wave. Here it is chosen as,

$$\alpha(x) = \frac{2}{P(x)},\ P(x) = (A_1A_0 - A_2e^{A_3x})^2(2\cos(3x/4) + 6\sin(5x/4) + 9). \tag{14}$$

The potentials $\alpha(x)$ and $W(x)$ are shown in figure 3a, when using (14). By substituting (12) into (11), we get

$$\varphi_1(x) = A_2 A_3 A_4 e^{A_3 x} \frac{\sqrt{\alpha(x)}}{P_0(x)} \cdot$$
$$(aA_1A_0 - aA_2 e^{A_3 x} + A_2 A_3 A_0 \sqrt{c_0} e^{A_3 x} sech(A_5 - A_4 \sqrt{c_0} h(x))),$$
$$P_0 = 2(A_1 A_0 - A_2 e^{A_3 x})^2 \beta (a + A_0 \sqrt{c_0} sech(A_5 - A_4 \sqrt{c_0} h(x))), \tag{15}$$

$$\varphi_2(x) = e^{-A_3 x} \frac{\sqrt{\alpha(x)}}{P_1(x)} \cdot$$
$$(aA_1A_0 - aA_2 e^{A_3 x} + A_2 A_3 A_0 \sqrt{c_0} e^{A_3 x} sech(A_5 - A_4 \sqrt{c_0} h(x))),$$
$$P_1(x) = A_2 A_3 A_4 (a + A_0 \sqrt{c_0} sech(A_5 - A_4 \sqrt{c_0} h(x))),$$

$$h(x) = \int \frac{dx}{(A_1 A_0 - A_2 e^{A_3 x})^2 \alpha(x)}, a = \sqrt{c_1^2 - 4c_0 c_2}. \tag{16}$$

Results in (15) and (16) are used to display $Re\psi_i$, $i = 1, 2$ against x and t for some numerical values of the free parameters, in figures 3b and 3c. Figure 3b displays periodic waves in time and oscillatory waves with cascade in the space, while figure 3c shows multi-periodic waves in the time and the space.

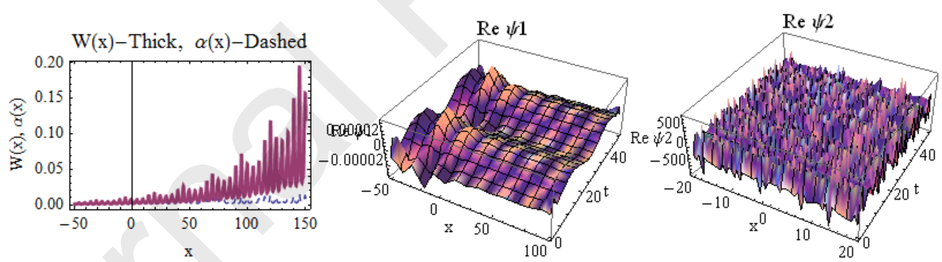


Figure 3. (a) shows the potentials $\alpha(x)$ (dashed) and $W(x)$ (thick). (b) and (c) 3D plots of $Re\psi_1$ and $Re\psi_2$. The parameters are: $A_0 = 1.5, A_1 = 3.5, A_2 = -A_1 A_0 A_3 := -0.01, \sigma = 1, A_4 = 0.7, \mu = 2, \beta = 1.5, a := 1.5, c_0 := 1.5, A_5 := 0$).

We remark that (b) shows a cascade of solitons wave while (c) sows multi periodic waves in space and time. this shows a variety of optical waves structures.

Case III. When $p=2$, $k=1$, we have

$$\begin{array}{c}\varphi_1(x)=\frac{a_1(x)\,g(x)+a_0(x)}{s_1(x)\,g(x)+s_0(x)},\ \varphi_2(x)=\frac{b_1(x)\,g(x)+b_0(x)}{s_1(x)\,g(x)+s_0(x)},\\ g'(x)=q(x)\,\sqrt{(c_2g(x)^2+c_1g(x)+c_0)}\end{array}. \tag{17}$$

By substituting (17) into (3), one obtains

$$\begin{array}{c}\beta=3\sigma,\ c_2=1/3(9A_1^2-14A_1B_0+9B_0^2)\sigma,\\ a=-\frac{\sqrt{6}A_0^3\sigma(A_1+B_0)^2\sqrt{-3A_1^2+10A_1B_0-3B_0^2}}{A_0^4(A_1+B_0)},\\ c_1=\frac{(9A_1^2-46A_1B_0+9B_0^2)\sigma}{3A_0},\ A=\frac{A_1A_0(B_1+2A_0B_0)}{(4B_1-AoBo)},\\ B_1=\frac{1}{4}A_0(-3A_1+B_0),\ A_1=\frac{(43+12\sqrt{7})B_0,}{29}\end{array}$$

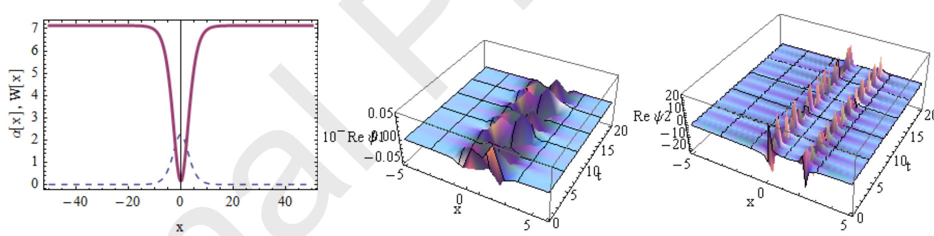
$$\begin{array}{c}a_1(x)=\frac{(-11+3\sqrt{7})\sqrt{43-12\sqrt{7}}s_0[x]\sqrt{\alpha(x)}}{58\sqrt{3}\sqrt{\sigma}},\\ a_0(x)=\frac{((43+12\sqrt{7})^{3/2}\sqrt{(2857-1032\sqrt{7}}s_0(x)\sqrt{\alpha(x)}}{1682\sqrt{3}\sqrt{\sigma}},\\ b_1(x)=-\frac{(((25+9\sqrt{7})A_0\sqrt{43-12\sqrt{7}}s_0[x]\sqrt{\alpha(x)})}{58\sqrt{3}\sqrt{\sigma}},\\ b_0(x)=\frac{(\sqrt{43-12\sqrt{7}}s_0[x]\sqrt{\alpha(x)})}{2\sqrt{3}\sqrt{\sigma}},\\ q(x)=\frac{A_0\sqrt{43-12\sqrt{7}}\sqrt{\alpha(x)})}{2\sqrt{3}\sqrt{\sigma}},\ W(x)=\mu-\frac{82\alpha(x)}{27}+\frac{A_0^2\alpha'(x)^2}{16\backslash\alpha(x)^2}.\end{array} \tag{18}$$

$$g(x)=-\frac{(-70+\frac{972e^{-2\sqrt{23}h(x)///9}}{C_2}+C_2e^{2\sqrt{23}h(x)///9})}{92A_0},\ h(x)=\int\alpha(x)dx. \tag{19}$$

By substituting (18), (19) and (20) into (17), one finds

$$
\begin{aligned}
&\varphi_1(x) = \tfrac{R_1(x)}{R_2(x)},\ \varphi_2(x) = -\tfrac{S_1(x)}{S_2(x)},\\
&R_1(x) = \sqrt{\alpha(x)(43-12\sqrt{7})}(972(-11+3\sqrt{7}) - 18(177+73\sqrt{7}C_2e^{2\sqrt{23}h(x)/9}\\
&\quad +(-11+3\sqrt{7})C_2^2e^{4\sqrt{23}h(x)/9})),\\
&R_2(x) = 58\sqrt{3}((972-162e^{-2\sqrt{23}h(x)/9}) + C_2^2e^{4\sqrt{23}h(x)/9})\sqrt{\sigma},\\
&S_1(x) = 46\sqrt{\alpha(x)(43-12\sqrt{7})}(-29-(\tfrac{25+9\sqrt{7}}{92})\cdot\\
&\quad (-70+\tfrac{972e^{-2\sqrt{23}h(x)/9}}{C_2} + C_2e^{2\sqrt{23}h(x)/9}),\\
&S_2(x) = 29\sqrt{3}(162 - \tfrac{(972e^{-2\sqrt{23}h(x)/9})}{C_2} - C_2e^{2\sqrt{23}h(x)/9})\sqrt{\sigma},
\end{aligned}
\tag{20}
$$

where $\alpha(x)$ is still a free function, which is taken here as a solitary wave, $\alpha(x) = {k_0}^2\text{sech}(rx)^2$. We mention that the two potentials $\alpha(x)$ and $W(x)$ are solitary (bright) and anti-solitary (dark) waves. They play the role of a potential wall and a potential well respectively, and they are shown in figure 4a. The results in (20) are displayed against x and t in figures 4b and 4c.



Figures 4a, 4b, and 4c. The potentials, and the 3D plots of $Re\psi_1$ and $Re\psi_2$.
The parameter values are
$A_0 = 3, k_0 = 1.5, r = 1/4, C_2 = 5, \mu = 7, \sigma = 2.5,\ \beta = 3\sigma.$

These figures show the propagation of two vectors of solitons which are localized in the space, with a pulse separator within each waveguide. Which may be interpreted as being due to the potential well. We remark that no wave propagation is visible outside the well, when $\mid x \mid > 20$.

4 The Eigenvalue problem and the stability

Here, to study the stability of waves, we consider the corresponding eigenvalue problem, by making a perturbation in the frequency,

$$\psi_j = e^{-i\mu t}\varphi_j + \varepsilon_j e^{\lambda t - i\mu t}(u_j + iv_j),\ j = 1, 2. \tag{21}$$

By substituting (21) into (1) one gets a set of equations that may be written in the matrix form:

$$P\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} = 0,\ Q\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} = 0,\ P = \begin{pmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{pmatrix}, Q = \begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}, \tag{22}$$

where the matrix elements are

$$\begin{gathered}
p_{11} = -\mu u_1[x] + \lambda v_1(x) + u_1(x)W[x] + 2\sigma u_1(x)\varphi_1(x)^2 + \tfrac{1}{2}u_1''(x],\\
p_{12} = -u_2(x)\,\alpha(x) + 2\beta u_2(x)\varphi_1(x)\varphi_2(x),\\
p_{21} = -\lambda u_1(x) - \mu v_1(x) + v_1[x]W(x) + 2\sigma v_1(x)\varphi_1(x)^2 + \tfrac{1}{2}v_1''(x)\\
p_{22} = -v_2[x]\alpha(x) + 2\beta v_2[x]\varphi_1(x)\backslash\varphi_2(x),
\end{gathered} \tag{23}$$

and

$$\begin{gathered}
q_{11} = -u_1(x)\alpha(x) + 2\beta u_1(x)\varphi_1(x)\varphi_2(x),\\
q_{12} = -\mu u_2(x) + \lambda v_2(x) + u_2(x)W(x) + 2\sigma u_2(x)\varphi_2(x)^2 + \tfrac{1}{2}u_2''(x),\\
q_{21} = -v_1(x)\alpha(x) + 2\beta v_1(x)\varphi(x)\varphi_2(x),\\
q_{22} = -\lambda u_2(x) - \mu v_2(x) + v_2(x)W(x) + 2\sigma v_2(x)\varphi_2(x)^2 + \tfrac{1}{2}v_2''(x).
\end{gathered} \tag{24}$$

Equations (22) hold when $\det(P) = 0$ and $\det(Q) = 0$; these equations give rise to the relations

$$\begin{array}{c}(2u_1(x)\,(\lambda u_2(x) + v_2(x)(-\mu + W(x) + 2\sigma\varphi_1(x)^2) \\ +v_2(x)\,(2\,\lambda v_1(x) + u_1''(x) + u_2(x)\,(2v_1(x)(\mu - W(x) - 2\sigma\varphi_1(x)^2) - v_1''(x)) = 0,\end{array} \tag{25}$$

and

$$\begin{array}{c}(v_1(x)(2\lambda v_2(x) + u_2(x)(-2\mu + 2W(x) + 4\sigma\varphi_2(x)^2 \\ +u_2''(x)) + u_1(x)(2\,\lambda u_2(x) + 2v_2(x)\,(\mu - W(x) - 2\sigma\varphi_2(x)^2 - v_2''(x))\end{array}. \tag{26}$$

Thus, the solution of the eigenvalue problem is reduced to finding solutions of (25) and (26), subject to boundary conditions $u_i(\pm\infty) = 0,\ v_i(\pm\infty) = 0,\ i = 1, 2.$

It is worth noticing that $\lambda = 0$ is an eigenvalue with eigenfunctions

$$\begin{array}{c}u_1(x) = B_1\varphi_1'(x),\ v_1(x) = B_2\varphi_1'(x), \\ u_2(x) := B_3\varphi_2'(x),\ v_2(x) = B_4\varphi_2'(x),\ \varphi_i'(\pm\infty = 0),\ i = 1, 2.\end{array} \tag{27}$$

This conclusion holds also from the topological invariance of solutions. It suggests to write the eigen functions when $\lambda\epsilon\mathbb{C}$ as

$$\begin{array}{c}u_1(x) = B_1\varphi_1'(x)\,(1 + \sum_{k=1}^{\infty} p_1^k\lambda^k\varphi_1(x)^k), \\ v_1(x) = B_2\varphi_1'(x)\,(1 + \sum_{k=1}^{\infty} p_2^k\lambda^k\varphi_1(x)^k), \\ u_2(x) = B_3\varphi_2'(x)\,(1 + \sum_{k=1}^{\infty} p_3^k\lambda^k\varphi_2(x)^k), \\ v_2(x) = B_4\varphi_2'(x)\ (1 + \sum_{k=1}^{\infty} p_4^k\lambda^k\varphi_2(x)^k).\end{array} \tag{28}$$

We mention that the eigenfunctions in (28) are not unique. But we assume that by the topological invariance, the stability is invariant. For simplicity we take the case when $k = 1$ in (28). One finds that

$$\begin{aligned} u_1(x) &= B_1\varphi_1'(x)\,(1+p_1\varphi_1(x)),\ u_2(x) = B_3\varphi_1'(x)\,(1+p_3\varphi_2(x)), \\ v_1(x) &= B_2\varphi_1'(x)\,(1+p_2\varphi_1(x)),\ v_2(x) = B_4\varphi_1'(x)\,(1+p_4\varphi_2(x)) \end{aligned} \tag{29}$$

$$v_1(x) = \frac{B_2 u_1(1+p_2\lambda\varphi_1(x))}{B_1(1+p_1\lambda\varphi_1(x))}, \quad v_2(x) = \frac{B_4 u_2(1+p_4\lambda\varphi_2(x))}{B_3(1+p_3\lambda\varphi_2(x))}. \tag{30}$$

By substituting (29) into (25) and (26), we get two lengthy equations of the form

$$u_1''(x) = f_1(\varphi_1(x), ..., \varphi_1''(x)), \;\; u_2''(x) = f_2(\varphi_2(x), ..., \varphi_2''(x), \tag{31}$$

which are not reproduced explicitly here.

5 Applications

Now, we consider a few examples of the stability analysis.

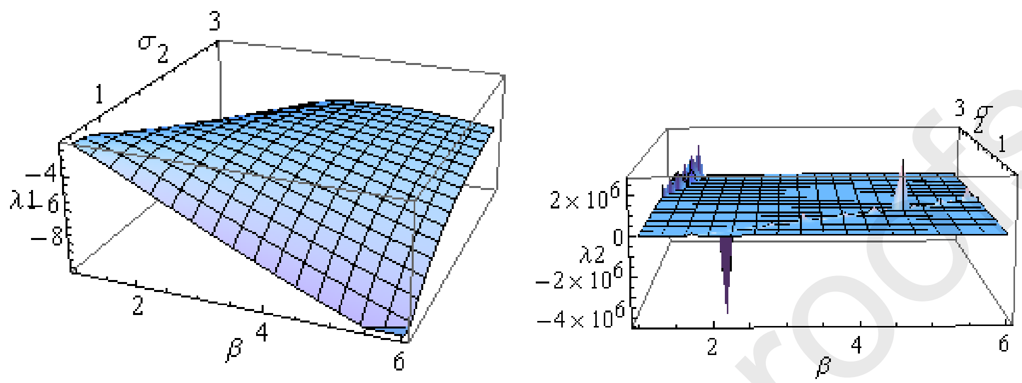
(i) First, we find the eigenvalues of the solutions given in (10). By using the equation $\varphi_2(x) = -\varphi_1(x)$ we have

$$u_2(x) = -\frac{B_3 u_1(x)\,(1+p_3\lambda\varphi_1(x))}{B_1\,(1+p_1\lambda\varphi_1(x)}. \tag{32}$$

By substituting (31) into (25) and (26), they reduce to two equations for $u_1''(x)$. They are consistent with each other when

$$B_4 = -B_1B_3/B_2,\ p_4 = p_2,\ p_3 = p_1. \tag{33}$$

By substituting (32), into the equation for $u_1''(x)$ and by integrating over $(-\infty, \infty)$, the eigenvalues λ can be found. We get two eigenvalues, namely λ_1 and λ_2, which are displayed against β and σ in figures 5a and 5b.



Figures 5a and 5b. Eigenvalues for the parameters
$c_1 := 0.2, c_0 = -0.09, C_1 := 0.15, C = -0.2, k_0 := 0.6, A_0 = 0.2, \mu = 1.3, p_1 = 5; p_2 := 3; B_1 = 2.5, B = 1.5.$

Figure 5a shows that $\lambda_1 \leq 0$, while 5b shows

$$\begin{cases} \lambda_2 > 0, & \beta < 1, \text{ and } (\beta, \sigma) \epsilon \mathbb{R} \\ \lambda_2 \leq 0 & \text{otherwise} \end{cases}, \tag{34}$$

where C is the characteristic curve shown in Fig 5b , $2 \leq \beta \leq 6$. Thus, the solutions in (10) are unstable (or stable) in the domains mentioned in (34).

(ii) Next, we consider the wave function solutions in (15), and find the eigenvalues in this case. The procedure is similar to the previous example. We start by finding the equation for $u_2(x)$ and $u_1(x)$, by using (15),

$$\begin{gathered} \varphi_2(x) = \frac{(2\,(A_1 A_0 e^{-A_3 x} - A_2)\beta\,\varphi_1(x)}{A_2^2 A_3^2 A_4^2}, \\ u_2(x) = \frac{2B_2\beta\,(1+\lambda\varphi_2(x))(A_1 A_0 e^{-A_3 x} - A_2)}{A_2^2 A_3^2 A_4^2}). \\ (\frac{(e^{-A_3 x} - A_2)u_1(x)}{B_1\,(1+\lambda\varphi_2(x))} - 4A_1 A_0 e^{-A_3 x}\varphi_1(x)). \end{gathered} \tag{35}$$

By substituting (35) into (25) and (26), one gets two equations for $u_1''(x)$. By solving the first one for $u_1''(x)$, one obtains a lengthy equation that is not reproduced here. By substituting this in the second equation, one finds the consistency condition

$$p_3 = p_2, B_4 = B_2 B_3/B1, p_1 := p_2, p_4 := p_2 B_1 := iB_2. \tag{36}$$

Using (36) one finds

$$u_1''(x) = 2u_1(x)(i\,\lambda + \mu - W(x) - 2\sigma\,\varphi_1^2(x)). \tag{37}$$

By importing (15) and (29) into (37) one obtains a lengthy equation. By integrating this equation over $(-\infty, \infty)$ and solving for λ, one finds that the eigenvalues are complex. It has been found that $Re\{\lambda_1\} > 0$ and $Re\{\lambda_2\} < 0$, so that the waves are unstable in this case.

6 Discussion

It is worthy to mention that our aim here is focused on optical surface structures, while experimental works were focused on BECs [37,38] or super fluids [39]. In this respect we mention that the existence of vortexes requires 3D helical potential or at least two dimensional problem. thus the comparison with experimental works does not hold.

In comparison with recent work done in [30] where the equation (1) was considered but for a single external potential $W(x)$ but $\alpha(x)$ was taken equal to constant κ.The homotopy renormalization mthod was used and the only results found there are solitary and coupled solitary waves where asymptotic solutions were obtained. We mention that here and in [30] attention was focused to stationary solutions. Here we found many different optical surface structures. Also, the eigen value problem is studied via finding the spectrum but this was not considered in [30]

On the other side the conditions for the existence of the rogue waves are not satisfied in the present case [40], as far as we are concerned with finding stationary (periodic) solutions which results to a vector of solitons with different structures. Indeed, the rogue waves arise currently in systems of higher differentiable order (> 2) or of dimension (> 1).

7 Conclusions

In this paper, we have considered coupled nonlinear Schrodinger equations of the Gross-Pitaevskii type, with linear mixing and nonlinear cross-phase modulation. For this system, we have obtained different kinds of wave structures induced by two injected beams, of physical relevance in nonlinear optics and BECs. Exact solutions are found by employing the extended unified method. It was shown that the geometrical waves structures depend on the type of external potentials that may take solitary, double hump solitary, Gaussian or some other oscillatory wave form.Many optical wae structures are shown as doubly wave vectors, cascade of waves and multi-periodic waves.

The eigenvalue problem for wave stability has been formulated. The eigenvalues are obtained for a few examples, and it was found that the stability (and instability) holds in specific domains in the (β, σ) plane. Wave cascades, oscillatory or trains of solitons are visualized. The analysis of their stability leads to the possibility of controlling the coupling of the whole wave, by means of changing the input power of the double beams. The results obtained are of general interest in nonlinear optics and Bose-Einstein condensation.

ACKNOWLEDGEMENTS

Thanks to the anonymous reviewers for their constructive comments for improving my manuscript.

The research work of the third author (MRB) was supported by the grant NPRP 8-028-1-001 from QNRF and he is thankful for it.

Compliance with ethical standards Conflict of interest: The authors declare that there were no conflicts of interest.

References

[1] Kevrekidis PG Frantzeskakis DJ, Carretero-Gonzalez R. Emergent nonlinear phenomena in Bose-Einstein condensates: theory and experiment. (2007) Berlin, Germany: Springer.

[2] Becker C, Stellmer S, Soltan-Panahi P, Dorscher S, Baumert M, Richter E-M, Kronjï¿œger J, Bongs K, Sengstock K. Oscillations and interactions of dark and dark-bright solitons in Bose-Einstein condensates. Nat. Phys. 4, (2008) 496–501.

[3] Kanna T, Lakshmanan M. Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrodinger equations, Phys. Rev. Lett. 86 (2001) 5043–5046.

[4] Vijayajayanthi M, Kanna T, Lakshmanan M. Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrodinger equations. Phys. Rev. A 77 (2008) 013820.

[5] Baronio F, Degasperis A, Conforti M, Wabnitz S. Solutions of the vector nonlinear Schrodinger equations: evidence for deterministic rogue waves. Phys. Rev. Lett. 109 (2012) 044102

[6] Yan Z, Dai C. Optical rogue waves in the generalized inhomogeneous higher-order nonlinear Schrodinger equation with modulating coefficients. J. Opt. 15, (2013) 064012.

[7] Yan Z. 2015 Integrable PT-symmetric local and nonlocal vector nonlinear Schrodinger equations: a unified two-parameter model. Appl. Math. Lett. 47, 61–68. (2015)

[8] Yan Z. Nonlocal general vector nonlinear Schrodinger equations: integrability, PT symmetribility, and solutions. Appl. Math. Lett. 62 (2016) 101–109.

[9] C. Weilnau, M. Ahles, J. Petter, D. Trager, J. Schroder, and C. Denz, Spatial optical (2+1)-dimensional scalar and vector solitons in saturable nonlinear media, Ann. Phys. 11(8), (2002) 573–629 .

[10] Songming Y.and Tian M 2017 The Gross-Pitaevskii Model of Spinor BEC, J. Appl. Math. (2012), Article ID 757616.

[11] Zhang G, Yan Z, Wen XY, Chen Y. Interactions of localized wave structures and dynamics in the defocusing coupled nonlinear Schrodinger equations. Phys. Rev. E 95 (2017) 042201.

[12] W. Bao, P.A. Markowich, C. Schmeiser, R.M. Weishaupl, On the Gross–Pitaevskii equation with strongly an isotropic confinement: formal asymptotic and numerical experiments, Math. Mod. Meth. Appl. Sci. 15 (2005) 67–782 .

[13] Torres P. J., Guided waves in a multi-layered optical structure, Nonlinearity 19 (2006) 2103-2113.

[14] A. I. Yakimenko, Y. A. Zaliznyak, and V. M. Lashkin, Two-dimensional nonlinear vector states in Bose- Einstein condensates, Phys. Rev. A 79(4) (2009) 043629.

[15] Torres P. J., Guided waves in a multi-layered optical structure, Nonlinearity 19 (2006) 2103-2113.

[16] W. Bao, Y. Zhang, Dynamical laws of the coupled Gross–Pitaevskii equations for spin-1 Bose–Einstein condensates, Methods Appl. Anal. 17 (2010) 49–80. .

[17] Juan Chen, Lu-ming Zhang, Numerical approximation of solution for the coupled nonlinear Schrodinger equations, Acta Mathematicae Applicatae Sinica, English Series, 33 (2017) 435–450|.

[18] L. P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961); E.P.Gross, Nuovo Cimento 20, 454 (1961); J.Math.Phys.4 (1963)4704 .

[19] Liu W., Geng X. G.and Xue B., Some generalized coupled nonlinear Schrodinger equations and conservation laws, Modern Physics Letters B Vol. 31, (2017) 1750299.

[20] Baronio F, Degasperis A, Conforti M and Wabnitz, Solutions of the Vector Nonlinear Schrodinger Equations: Evidence for Deterministic Rogue Waves, Phys. Rev. Lett. 109 (2012) 044102.

[21] Wang X, Yang B, Chen Y and Yang Y Q,,Higher-Order Localized Waves in Coupled Nonlinear Schrodinger Equations, Chin. Phys. Lett. 31 (2014) 090201.

[22] Abdel-Gawad H.I., Tantawy M., Mixed-type soliton-propagation in two-layer-liquid (or in an elastic) medium with waveguides dispersion, J.of Molec. Liqu, 241 (2017) 870–874.

[23] Wang X, Li Y Q and Chen Y, Generalized Darboux transformation and localized waves in coupled Hirota equations, Wave Motion 51 (2014)1149.

[24] Abdel-Gawad H. I.and Tantawy M., On N-mixed-type soliton propagation in non autonomous long waves dispersion with waveguides, Nonlinear Dyn (2017) 90:233–239.

[25] Zhao L-C. and Liu J., Rogue-wave solutions of a three-component coupled nonlinear Schrodinger equation, Phys. Rev. E 87, (2013) 013201.

[26] Bansala. A, Biswas, A, Mahmoode. M. F. , Zhouf. Q., Mirzazadehg. M., Alshomranic. A. S.,Moshokoad. S. P., Belic. M., Optical soliton perturbation with Radhakrishnan–Kundu– Lakshmanan equation by Lie group analysis.,Optik 163 (2017) 137–141.

[27] Abdel-Gawad. H. I.,On the $\hat{k}$ p -operator", new extension of the KdV6 to (m + 1)-dimensional equation and traveling waves solutions.Nonlinear Dyn. **85** (2016) 895

[28] Abdel-Gawad, H.I., Biswas, A.: Multi-soliton solutions based on interactions of basic traveling waves with an applications to the nonlocal Boussinesq equation. Acta Phys. Pol. B 47 (2016) 1101–1112.

[29] Zafrany A., Malomed B. A., and Merhasin I. M., Solitons in a linearly coupled system with separated dispersion and nonlinearity, Chaos 15, (2005) 037108.

[30] Yue Kai, Exact solutions and asymptotic solutions of one-dimensional domain walls in nonlinearly coupled system, Nonlinear Dyn. 92: (2018) 1665–1677.

[31] Abdel-Gawad, H. I., and Tantawy M. , M.: On controlled propagation of long waves in nonautonomous Boussinesq–Burgers equations. Nonlinear Dyn. **87** (2017) 2511–2518.

[32] Abdel-Gawad, H. I., El-Azab, N. and Osman M., Exact solution of the space -dependent KdV equation, JPSP ,, 82 (2013) 044004,

[33] Abdel-Gawad, H.I. and Osman M., Exact solutions of the KdV equation with space and tme dependent coefficients by the extended unified method. Indian J. Pure Appl. Math., 45 (2014) 1-11.

[34] Abdel-Gawad, H.I., and Tantawy M. , M.: Waveguides of two-soliton solutions for the coupled KdV equations with variable coefficients in

long-distance communication systems, Indian J Phys **91**(6) (2017) 671–675.

[35] Bodenschatz, E., Pesch, W., Ahlers, G.: Recent developments in Rayleigh–Bénard convection. Annu. Rev. Fluids Mech. 32 (2000) 709–778,

[36] Dror, N., Malomed, B.A., Zeng, J. , Domain walls and vortices in linearly coupled systems. Phys. Rev. E 84, (2011) 046602.

[37] .Hensinger W.K., Haffner H., Browaeys A.,. Heckenberg N. R.,Helmerson K.and McKenzie C.., Nature(Vol. 412, Issue 6842.). Dynamical tunnelling of ultracold atoms Citation metadata , Nature, 412, no. 6842, (2001), p. 52.

[38] Hensinger W.K., ∗ A. Mouche A., Julienne P. S., Delande D., Heckenberg N. R. and H. Rubinsztein D. H., Analysis of dynamical tunnelling experiments with a Bose-Einstein condensate., Phys Rev A.70. (2004) 013408.

[39] Shukla V., Pandit R., Brachet M.,Particles and Fields in Superfluids: Insights from the Two-dimensional Gross-Pitaevskii Equation.,Phys. Rev. A 97 (2018) 013627.

[40] Abdel-Gawad H.I.,Tantawy M. and Abou Elkhair R. E.,On the extension of solutions of the real to complex KdV equation and a mechanism for the construction of rogue waves., Waves in Random and Complex Media, 26 (2016) 397-406.

1- In this paper, we investigate the construction of optical surface structures.

2- To this end, we have studied the coupled non-linear Schrodinger equations with double external potentials.

3- We have shown many different structures; cascade-waves, doubly periodic waves, and vector soliton waves.

4- The method used in this study is the extended unified method where white-class of self-similar solutions and traveling waves solutions are found.