

Quantum Fluctuations and New Instantons II: Quartic Unbounded Potential

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We study the fate of a false vacuum in the case of a potential that contains a portion which is quartic and unbounded. We first prove that an $O(4)$ invariant instanton with the Coleman boundary conditions does not exist in this case. This, however, does not imply that the false vacuum does not decay. We show how the quantum fluctuations may regularize the singular classical solutions. This gives rise to a new class of $O(4)$ invariant regularized instantons which describe the vacuum instability in the absence of the Coleman instanton. We derive the corresponding solutions and calculate the decay rate they induce.

1. Introduction

A false vacuum, corresponding to a local minimum of the scalar field potential $V(\varphi)$, is unstable and decays via tunneling. The main contribution to the decay rate is suggested to be given by the symmetric $O(4)$ - solutions for the scalar field in Euclidean time, called instantons, which satisfy the equation^[1]:

$$\ddot{\varphi}(\varphi) + \frac{3}{\varphi} \dot{\varphi}(\varphi) - \frac{dV}{d\varphi} = 0, \quad (1)$$

where a dot denotes the derivative with respect to $\varrho = \sqrt{\tau^2 + \mathbf{x}^2}$, $\tau = it$ is the Euclidean time and $\mathbf{x} = (x^1, x^2, x^3)$ are the spatial coordinates. At the instant $\tau = 0$, the instanton emerges from under

the barrier as a bubble of size ϱ_0 and expands. The probability of the decay per unit time and unit volume can then be estimated as

$$\Gamma \simeq \varrho_0^{-4} \exp(-S_I), \quad (2)$$

where

$$S_I = 2\pi^2 \int_0^{+\infty} d\varrho \varrho^3 \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) \quad (3)$$

is the instanton action (for details see [1] and, for example, [2–4]). Equation (1) is an ordinary second order differential equation and requires two boundary conditions, which according to [1] must be taken as

$$\varphi(\varrho \rightarrow \infty) = \varphi_0, \quad (4)$$

where φ_0 is the local false minimum of $V(\varphi)$, and

$$\dot{\varphi}(\varrho = 0) = 0. \quad (5)$$

The second condition is imposed to avoid a singularity in the “center of the bubble” and to assure a finite instanton action. The condition (5), formulated in the deep ultraviolet region ($\varrho \rightarrow 0$), reduces the infinite number of $O(4)$ invariant solutions of equation (1) to a single non-singular solution. This, however presents challenges when describing the false vacuum decay for a broad class of potentials.

In particular, in [4] we have demonstrated that for the case of a potential with a false vacuum and a steep unbounded linear portion, the usage of the Coleman boundary conditions ad littera leads to questionable results. For example, in the case of a very steep unbounded potential the Coleman instanton leads to unexpectedly high decay rate via small size instantons. It was shown that this problem is resolved once one recognizes the inevitable effects of the quantum fluctuations, which impose both the ultraviolet and the infrared cutoffs on the range of the validity of classical solution. In particular, the presence of the ultraviolet cutoff, determined by the parameters of the classical solution, allows us to abandon the need to implement condition (5) as the regularization has removed the potential singularity. As a result we obtain a new class of $O(4)$ invariant regularized solutions each of which contributes to the decay rate.

In this paper we apply these considerations to a class of potentials that contain both a false vacuum and a portion of an unbounded quartic potential. This potential is nearer in form to that in the standard model. First we prove that the instantons with the Coleman boundary conditions do not exist in this case, which in turn could lead to the erroneous conclusion that the false vacuum

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4. Quantum Fluctuations

As we argued in [4] the classical instanton can be trusted only in that region where the contribution of the classical field and its derivative exceed the level of minimal quantum fluctuations in the corresponding scales, which can be roughly estimated to be (see, for example, [7]):

$$|\delta\varphi_q| \simeq \frac{\sigma}{\rho}, \quad |\delta\dot{\varphi}_q| \simeq \frac{\sigma}{\rho^2}, \quad (16)$$

where σ is the numerical coefficient of order unity (we use $\hbar = 1$ units). For quantum fluctuations $\rho^4 \delta\dot{\varphi}_q^2 \simeq \sigma^2$ and $|\rho^3 \delta\dot{\varphi}_q \delta\varphi_q| \simeq \sigma^2$, and, therefore, as it follows from (15), for $\varphi < \varphi_m$ the classical instanton solution makes sense only if

$$E_- > 3\sigma^2. \quad (17)$$

The corresponding solution with positive E_- is given in terms of Jacobi elliptic functions and diverges as $\rho \rightarrow 0$. However, as we have shown in [4], this solution $\varphi(\rho)$ is valid only at $\rho > \rho_{uv}$ where ρ_{uv} is the ultraviolet cutoff scale determined by the condition

$$\dot{\varphi}(\rho_{uv}) \simeq \frac{\sigma}{\rho_{uv}^2}. \quad (18)$$

The solution of this equation for ultraviolet cutoff is presented in the next section. This cutoff regularizes the instantons which are singular otherwise and have infinite action.

5. New Instantons

The calculations leading to the explicit form of the $O(4)$ invariant regularized solution are somewhat more involved in the quartic case compared to those in the unbounded linear behavior case.^[4] We now sketch the various steps involved. First one identifies the parameter on which the solution depends. One then proceeds to find the ultraviolet cutoff scale ρ_{uv} (the infrared cutoff will not play an important role) in terms of the parameters of the solution. Next we will construct the explicit regularized solutions valid in the range above the ultraviolet cutoff scale. Equipped with these solutions one determines the action and the false vacuum decay rate. This involves various Jacobi elliptic functions. Finally we will present the results for various asymptotic limits.

The ultraviolet cutoff scale regularizes a whole class of new instantons which can be parametrized either by E_- or alternatively by $\rho_0(E_-)$. It is convenient to parametrize them by a new variable χ related to ρ_0 in (10) as

$$\rho_0^2 \varphi_0^2 = \frac{E_c}{4(1-\beta)} \chi (1-\beta+\chi), \quad (19)$$

where

$$E_c \equiv \frac{32}{\lambda_+(1-\beta)}. \quad (20)$$

Substituting (19) in the expression for δ in (10) we find

$$\delta = \frac{1-\beta}{\chi}. \quad (21)$$

The solution (10) is valid only $\rho > \rho_m$, where ρ_m is determined by

$$\varphi(\rho_m) \equiv \varphi_m = \beta \varphi_0, \quad (22)$$

from where one gets

$$\rho_m^2 \varphi_0^2 = \frac{E_c}{4(1-\beta)} \chi (1+\chi). \quad (23)$$

At $\rho = \rho_m$ we have to match (10) to a solution of equation (11) with $E_- > 0$. This solution is given in terms of the Jacobi elliptic functions,^[8]

$$\varphi(\rho) = \frac{\sqrt{2}k}{\rho \sqrt{\lambda_- (2k^2 - 1)}} \operatorname{cn}\left(\frac{\ln(\rho/\alpha)}{\sqrt{2k^2 - 1}}, k\right), \quad (24)$$

where

$$k \equiv \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\sqrt{1 + 2\lambda_- E_-}}}, \quad \frac{1}{\sqrt{2}} < k \leq 1, \quad (25)$$

and the constants of integration α and E_- are determined in terms of ρ_0^2 or, alternatively χ , by requiring that the field φ and its first derivative are continuous at ρ_m .

First let us find $E_-(\chi)$. The expression (15) for E_- rewritten in terms of the field values at the matching point ρ_m becomes

$$E_- = \rho_m^4 \dot{\varphi}_m^2 + 2\rho_m^3 \dot{\varphi}_m \varphi_m + \frac{\lambda_-}{2} \rho_m^4 \varphi_m^4. \quad (26)$$

On the other hand, the first integral of equation (9) expressed in terms of the field values at the same point is equal

$$E_+ = \rho_m^4 \dot{\varphi}_m^2 + 2\rho_m^3 (\varphi_m - \varphi_0) \dot{\varphi}_m - \frac{\lambda_+}{2} \rho_m^4 (\varphi_m - \varphi_0)^4, \quad (27)$$

and it vanishes for the solution (10). Keeping this in mind and subtracting (27) from (26) we find

$$E_-(\chi) = E_c \chi (1+\chi)^3. \quad (28)$$

From here it follows that the quantum bound (17) imposes the following lower bound on the possible values of χ

$$\chi > \chi_c \simeq \frac{3\sigma^2}{E_c}. \quad (29)$$

If not, even with the best of intentions, the spirit of the semi-classical approximation can not be retained and the quantum effects take over. Because the field must be continuous at ρ_m the constant of integration α is determined by solving the equation we obtain after substitution (24) in (22):

$$\operatorname{cn}\left(\frac{\ln(\rho_m/\alpha)}{\sqrt{2k^2 - 1}}, k\right) = \sqrt{\lambda_- \beta^2 \rho_m^2 \varphi_0^2 \left(1 - \frac{1}{2k^2}\right)} \equiv \varepsilon_m, \quad (30)$$

$$\operatorname{sn}\left(\frac{\ln(\rho_m/\alpha)}{\sqrt{2k^2 - 1}}, k\right) < 0,$$

where both ϱ_m , and k can be expressed through χ . The classical solution (24) has a bounce at $\varrho_b < \varrho_m$, at which $\dot{\varphi}(\varrho_b) = 0$. Equating the derivative of (24) to zero, one gets the following equation for ϱ_b :

$$\begin{aligned} \operatorname{cn}\left(\frac{\ln(\varrho_b/\alpha)}{\sqrt{2k^2-1}}, k\right) &= -\left(\frac{1}{k^2}-1\right)^{\frac{1}{4}} \equiv -\varepsilon_b, \\ \operatorname{sn}\left(\frac{\ln(\varrho_b/\alpha)}{\sqrt{2k^2-1}}, k\right) &> 0. \end{aligned} \quad (31)$$

Let us notice the following useful relation between ε_m and ε_b , which appear in equations (30) and (31):

$$\varepsilon_m = f(\chi) \varepsilon_b, \quad (32)$$

where

$$f(\chi) = \left(\frac{\beta \chi}{1+\chi}\right)^{1/4} < 1. \quad (33)$$

The solution (24) fails before the bounce is reached. In fact, at $\varrho_{uv} > \varrho_b$, determined by (18), the quantum fluctuations become dominant and the bubble emerges from under the barrier. Introducing ε_{uv} , defined as

$$\varepsilon_{uv} \equiv -\operatorname{cn}\left(\frac{\ln(\varrho_{uv}/\alpha)}{\sqrt{2k^2-1}}, k\right) \quad \text{for} \quad \operatorname{sn}\left(\frac{\ln(\varrho_{uv}/\alpha)}{\sqrt{2k^2-1}}, k\right) > 0, \quad (34)$$

and substituting (24) in (18) we find that ε_{uv} satisfies the following equation

$$\left(\frac{\varepsilon_{uv}}{\varepsilon_b}\right)^4 - \left(\frac{32\sigma^4}{\lambda_- E_-^3}\right)^{1/4} \frac{\varepsilon_{uv}}{\varepsilon_b} - \left(1 - \frac{\sigma^2}{E_-}\right) = 0, \quad (35)$$

the exact solution of which is

$$\frac{\varepsilon_{uv}}{\varepsilon_b} = \left(1 - \frac{\sigma^2}{E_-}\right)^{1/4} W \left(\sqrt{1 - \frac{\gamma}{\sqrt{27} W^6}} + \sqrt{\frac{\gamma}{\sqrt{27} W^6}} \right), \quad (36)$$

where

$$\gamma = \sqrt{\frac{27\sigma^4}{8\lambda_- E_-^3} \left(1 - \frac{\sigma^2}{E_-}\right)^{-1}}, \quad (37)$$

and

$$W = \left[\frac{1}{3} \left(1 + \left(\sqrt{1+\gamma^2} + \gamma \right)^{2/3} + \left(\sqrt{1+\gamma^2} - \gamma \right)^{2/3} \right) \right]^{1/4}. \quad (38)$$

Now keeping in mind the signs of the Jacobi $\operatorname{sn}(x, k)$ in (30), (31) and (34) we first find that the elliptic Jacobi amplitudes are

$$\operatorname{am}\left(\frac{\ln(\varrho_m/\alpha)}{\sqrt{2k^2-1}}, k\right) = 2\pi - \operatorname{arccos}(\varepsilon_m),$$

$$\operatorname{am}\left(\frac{\ln(\varrho_{uv}/\alpha)}{\sqrt{2k^2-1}}, k\right) = \operatorname{arccos}(-\varepsilon_{uv}),$$

$$\operatorname{am}\left(\frac{\ln(\varrho_b/\alpha)}{\sqrt{2k^2-1}}, k\right) = \operatorname{arccos}(-\varepsilon_b), \quad (39)$$

and then obtain the following expressions for the ratio of the corresponding scales

$$\begin{aligned} \ln \frac{\varrho_m}{\varrho_b} &= \sqrt{2k^2-1} (2K(k) + F(\pi - \operatorname{arccos}(-\varepsilon_b), k) \\ &\quad - F(\operatorname{arccos}(\varepsilon_m), k)), \end{aligned} \quad (40)$$

and

$$\begin{aligned} \ln \frac{\varrho_m}{\varrho_{uv}} &= \sqrt{2k^2-1} (2K(k) + F(\pi - \operatorname{arccos}(-\varepsilon_{uv}), k) \\ &\quad - F(\operatorname{arccos}(\varepsilon_m), k)), \end{aligned} \quad (41)$$

where $F(x, k)$ and $K(k)$ are incomplete and complete elliptic integral of the first kind correspondingly.

Taking into account the definitions of ε_m and ε_{uv} in (30) and (34) we find from (24) the following useful expressions for the value of the field on the border of quantum core,

$$\varphi_{uv} \equiv \varphi(\varrho_{uv}) = -\frac{\varrho_m \varepsilon_{uv}}{\varrho_{uv} \varepsilon_m} \beta \varphi_0, \quad (42)$$

and for the potential

$$V_{uv} \equiv V(\varphi_{uv}) = -V_{bar} \left(\beta \left(\frac{\varrho_m \varepsilon_{uv}}{\varrho_{uv} \varepsilon_m} \right)^4 - 1 \right), \quad (43)$$

correspondingly.

Finally let us calculate the action for the regularized χ -instan- tons. Substituting (10) and (24) into

$$S_I = 2\pi^2 \int_{\varrho_{uv}}^{+\infty} d\varrho \varrho^3 \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) + \frac{\pi^2}{2} V(\varphi(\varrho_{uv})) \varrho_{uv}^4, \quad (44)$$

where the second term account for the contribution of quantum core (see, [4] for justification), we obtain

$$\begin{aligned} S_I(\chi) &= \frac{\pi^2}{3} \left[\frac{E_-}{4} \left(3 + \frac{1}{1+\chi} + \frac{1-2\beta}{(1-\beta)(1+\chi)^2} + \frac{1-\beta}{(1+\chi)^3} \right) \right. \\ &\quad + \frac{4(1+2\lambda_- E_-)^{1/4}}{\lambda_-} (4E(k) - E(\operatorname{arccos}(\varepsilon_m), k) \\ &\quad - E(\operatorname{arccos}(-\varepsilon_{uv}), k)) + \frac{E_-(2-3k^2)}{k^2} \ln \left(\frac{\varrho_m}{\varrho_{uv}} \right) \\ &\quad \left. + \left(\frac{2E_-}{\lambda_-} \right)^{1/2} \left(\frac{\varepsilon_{uv}}{\varepsilon_b} \right)^2 + 2\sigma \left(\frac{2E_-}{\lambda_-} \right)^{1/4} \frac{\varepsilon_{uv}}{\varepsilon_b} \right], \end{aligned} \quad (45)$$

where E_- can be expressed in terms of χ and vice versa using (28), ε_{uv} is the solution of equation (35) related to ϱ_{uv} as in (34), ε_b is

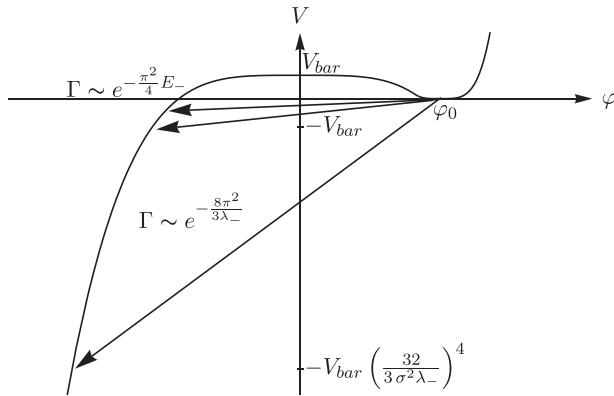


Figure 2. A potential in the case $\lambda_- \ll \lambda_+$. The values indicated on the lower part of vertical axis are the values of the potential at the core of the instanton. The corresponding values of the decay probability Γ are given in equation (64).

defined in (31), and $E(x, k)$ and $E(k)$ are incomplete and complete elliptic integral of the second kind, respectively.

6. Asymptotics

The exact solutions above are not very transparent and therefore it is useful to consider the limiting cases for which the special Jacobi functions simplify to elementary functions.

Let us begin with displaing the asymptotics of the exact solution (36), which contains the parameter γ , defined in (37). Because $\gamma \sim (\lambda_- E_-)^{-1/2}$, there are two cases we have to consider, namely, $\lambda_-^{-1/3} \gg E_- > 3\sigma^2$ and $E_- \gg \lambda_-^{-1/3}$, which correspond to $\gamma \gg 1$ and $\gamma \ll 1$ respectively. The asymptotics are

$$\frac{\epsilon_{uv}}{\epsilon_b} \simeq \left(\frac{32 \sigma^4}{\lambda_- E_-^3} \right)^{1/12} \quad \text{for} \quad \frac{1}{\lambda_-^{1/3}} \gg E_- > 3\sigma^2,$$

$$\simeq 1 + \left(\frac{\sigma^4}{8 \lambda_- E_-^3} \right)^{1/4} \quad \text{for} \quad E_- \gg \frac{1}{\lambda_-^{1/3}}. \quad (46)$$

As we see in the first case ϵ_{uv} can be significantly larger than ϵ_b . For example for the minimal possible $E_- \sim 3\sigma^2$, allowed by quantum limit, $\epsilon_{uv}/\epsilon_b \simeq (\sigma^2 \lambda_-)^{-1/12}$ can be larger than unity but only for very small λ_- . For $E_- \gg 1/\lambda_-^{1/3}$ the instant at which the classical description fails is always very close to the moment of the bounce.

There exist two qualitatively different classes of the potentials, namely, those for which $\lambda_+ \ll \lambda_-$ ($\beta \ll 1$) (see Figure 1) with rather sharp maximum and the potentials with $\lambda_- \ll \lambda_+$ and $1 - \beta \simeq (\lambda_-/\lambda_+)^{1/3} \ll 1$ (Figure 2) having rather flat maximum. We will consider these two cases separately.

The behavior of elliptic functions and integrals in the formulae above depends on the value of the parameter

$$2 \lambda_- E_- = 64 \frac{(1 - \beta)^2}{\beta^3} \chi (1 + \chi)^3. \quad (47)$$

For $2 \lambda_- E_- \ll 1$

$$k \simeq 1 - \frac{\lambda_- E_-}{4} \quad (48)$$

and both

$$\epsilon_b \simeq \left(\frac{\lambda_- E_-}{2} \right)^{1/4}, \quad (49)$$

and $\epsilon_m = f \epsilon_b$, where f is given in (33), are much smaller than unity, that is, $\epsilon_m, \epsilon_b \ll 1$. For $k \rightarrow 1$ the elliptic integrals diverge logarithmically. To reveal the main non-analytic term in the expansion of $F(\arccos(\epsilon), k)$ for $\epsilon \ll 1$ we first note that

$$\arccos(\epsilon) = \frac{\pi}{2} - \epsilon + O(\epsilon^3) \quad (50)$$

and

$$F(\arccos(\epsilon), k) = \int_0^{\frac{\pi}{2} - \epsilon + O(\epsilon^3)} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

$$= K(k) - \int_0^{\epsilon - O(\epsilon^3)} \frac{d\tilde{\alpha}}{\sqrt{1 - k^2 \cos^2 \tilde{\alpha}}}. \quad (51)$$

Keeping in mind that in the whole range of integration $\tilde{\alpha} \ll 1$ we can expand $\cos^2 \tilde{\alpha}$ in series and the integral in (51) can be approximated as

$$\int_0^{\epsilon - O(\epsilon^3)} \frac{1}{\sqrt{1 - k^2 + k^2 \tilde{\alpha}^2}} \left(1 + O\left(\frac{\tilde{\alpha}^4}{1 - k^2 + k^2 \tilde{\alpha}^2} \right) \right) d\tilde{\alpha}$$

$$= \frac{1}{k} \ln \frac{k \epsilon + \sqrt{1 - k^2 + k^2 \epsilon^2}}{\sqrt{1 - k^2}} + O(\epsilon^2), \quad (52)$$

and hence (51) becomes

$$F(\arccos(\epsilon), k) = K(k) - \frac{1}{k} \ln \frac{\epsilon + \sqrt{\epsilon_b^4 + \epsilon^2}}{\epsilon_b^2} + O(\epsilon^2), \quad (53)$$

where we took into account the definition of ϵ_b in (31). The complete elliptic integral $K(k)$ for $k = 1 - \epsilon$ has the following expansion (see [8])

$$K(k) = \frac{1}{2} \ln \frac{8}{\epsilon} + O(\epsilon \ln \epsilon). \quad (54)$$

For $2 \lambda_- E_- \gg 1$

$$k \simeq \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{8 \lambda_- E_-}} \right), \quad \epsilon_b \simeq 1 - \frac{1}{\sqrt{8 \lambda_- E_-}}. \quad (55)$$

The point $k = 1/\sqrt{2}$ is not singular and for example

$$K\left(\frac{1}{\sqrt{2}} + \epsilon \right) = \frac{1}{4\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4} \right) \right]^2 + O(\epsilon) \simeq 1.85 + O(\epsilon). \quad (56)$$

Using these expansions we can simplify the exact formulae derived in the previous section and obtain the expressions for V_{uv} , φ_0^2 and the action S as a function of E_- parametrizing the instantons. In the derivation one has to use E_- and χ interchangeably resolving (47) in the approximations $\chi \gg 1$ and $\chi \ll 1$. Skipping details of the calculations we will present below the results in the leading order.

The potential with a sharp maximum. First let us consider the potential shown in Figure 1 for which $\lambda_- \gg \lambda_+$ and, hence, $\beta \ll 1$. As it follows from (47) in this case

$$\begin{aligned} \chi &\simeq \frac{\lambda_+ E_-}{32} \quad \text{for } \frac{32}{\lambda_+} \gg E_- > 3\sigma^2, \\ &\simeq \left(\frac{\lambda_+ E_-}{32} \right)^{1/4} \quad \text{for } E_- \gg \frac{32}{\lambda_+} \end{aligned} \quad (57)$$

and substituting this in (19) we find

$$\begin{aligned} \varphi_0^2 \theta_0^2 &\simeq \frac{E_-}{4} \quad \text{for } \frac{32}{\lambda_+} \gg E_- > 3\sigma^2, \\ &\simeq \sqrt{\frac{2E_-}{\lambda_+}} \quad \text{for } E_- \gg \frac{32}{\lambda_+}. \end{aligned} \quad (58)$$

The calculation of V_{uv} is more involved. We have first to expand (40) and (41), using approximations derived above, and then with taking into account (32) and (46) from (43) one gets

$$\begin{aligned} V_{uv} &\simeq -V_{bar} \frac{\lambda_-}{\lambda_+} \left(\frac{32}{\lambda_- E_-} \right)^4 \quad \text{for } \frac{1}{2\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq -V_{bar} \left(\frac{32}{\lambda_+ E_-} \right) \quad \text{for } \frac{32}{\lambda_+} \gg E_- \gg \frac{1}{2\lambda_-}, \\ &\simeq -V_{bar} \left(\frac{32}{\lambda_+ E_-} \right)^{1/4} \quad \text{for } E_- \gg \frac{32}{\lambda_+}. \end{aligned} \quad (59)$$

Finally, one can check that the leading term in the action in different asymptotics is

$$\begin{aligned} S &\simeq \frac{8\pi^2}{3\lambda_-} \quad \text{for } \frac{1}{2\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq \frac{\pi^2}{2} E_- \quad \text{for } \frac{32}{\lambda_+} \gg E_- \gg \frac{1}{2\lambda_-}, \\ &\simeq \frac{\pi^2}{4} E_- \quad \text{for } E_- \gg \frac{32}{\lambda_+}. \end{aligned} \quad (60)$$

The potential with a rather flat barrier. In the case of a flat barrier (Figure 2) with $\lambda_- \ll \lambda_+$ and $1 - \beta \ll 1$ the calculations are very similar. The solution of (47) is

$$\begin{aligned} \chi &\simeq \frac{\lambda_- E_-}{32(1-\beta)^2} \quad \text{for } \frac{32(1-\beta)^2}{\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq \left(\frac{\lambda_- E_-}{32(1-\beta)^2} \right)^{1/4} \quad \text{for } E_- \gg \frac{32(1-\beta)^2}{\lambda_-}. \end{aligned} \quad (61)$$

Correspondingly, for the size of the bubble we obtain

$$\begin{aligned} \varphi_0^2 \theta_0^2 &\simeq \frac{E_-}{4} \left(1 + \frac{\lambda_+ E_-}{32} \right) \quad \text{for } \frac{32(1-\beta)^2}{\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq \sqrt{\frac{2E_-}{\lambda_-}} \quad \text{for } E_- \gg \frac{32(1-\beta)^2}{\lambda_-}, \end{aligned} \quad (62)$$

the potential V_{uv} is given by

$$\begin{aligned} V_{uv} &\simeq -V_{bar} \left(\frac{32}{\lambda_- E_-} \right)^4 \quad \text{for } \frac{1}{2\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq -V_{bar} \frac{12.45}{(\lambda_- E_-)^{1/4}} \quad \text{for } E_- \gg \frac{1}{2\lambda_-} \end{aligned} \quad (63)$$

and the action is

$$\begin{aligned} S &\simeq \frac{8\pi^2}{3\lambda_-} \quad \text{for } \frac{1}{2\lambda_-} \gg E_- > 3\sigma^2, \\ &\simeq \frac{\pi^2}{4} E_- \quad \text{for } E_- \gg \frac{1}{2\lambda_-}. \end{aligned} \quad (64)$$

In both cases the contribution of the E_- -instantons to the overall decay rate per unit volume per unit time is given by

$$\Gamma(E_-) \sim (\varphi_0(E_-))^{-4} e^{-S(E_-)}, \quad (65)$$

and from the formulae above one finds that for the unbounded potentials the main contribution comes from instantons with $1/2\lambda_- > E_- > 3\sigma^2$ for which the leading term in the action is $\frac{8\pi^2}{3\lambda_-}$. The contribution of these instantons only weakly depends on their size, which is always larger than φ_0^{-1} .

The results above can also be used to estimate the leading contribution to the decay rate for the potentials with two minima (see for example curve (a) in Figure 1). If the true minimum has the depth V_{min} , the corresponding E_- which gives the largest contribution to the decay rate is determined by the equation $V_{min} \simeq V_{uv}(E_-)$. For example, in the case $\lambda_+ \ll \lambda_-$ for the true minimum in the range

$$64 \frac{\lambda_-}{\lambda_+} V_{bar} \gg |V_{min}| \gg V_{bar} \quad (66)$$

from equations (58) and (59) we get

$$\Gamma \sim \left(\frac{\lambda_+ |V_{min}|}{8 V_{bar}} \right)^2 \varphi_0^4 \exp \left(-\frac{16\pi^2 V_{bar}}{\lambda_+ |V_{min}|} \right). \quad (67)$$

In the other cases the calculations are very similar.

7. Conclusions

In this paper we have considered the false vacuum decay for the unbounded quartic potential. Using exact solutions we have proven that the instanton with the Coleman boundary conditions does not exist in this case. One has to point out that this result does not contradict to the theorem in [5] because not all necessary conditions for the existence of solution with the Coleman boundary conditions are satisfied for quartic potentials. We suspect (although not proven this) that the Coleman instanton also does not exist for unbounded potential $-\varphi^n$ with $n > 4$. Taken literally this result, one could have erroneously concluded that the decay of the false vacuum cannot be described

anymore by the most symmetric $O(4)$ Euclidean solutions for these potentials.

In the previous paper^[4] we have shown that one of the boundary conditions used by Coleman to avoid the singularity of the classical solution must be abandoned because it is formulated in the deep ultraviolet region where quantum fluctuations definitely dominate and the classical solution cannot be trusted anymore. The quantum fluctuations naturally regularize the singular classical solutions by introducing the cutoff scale entirely determined by the parameters of the classical solution. In turn, this leads to the appearance of a new class of instantons, which being singular in the absence of cutoff, were not contributing to the false vacuum decay before. The new $O(4)$ -instantons allow us to calculate the leading contribution to vacuum decay rate in those case when the $O(4)$ -instantons with the Coleman boundary conditions do not exist. For the particular example of quartic unbounded potential we have found the explicit solutions, which can also be applied to a broad class of potentials with false and true minima separated by quartic potential.

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Conflict of Interest

The authors have declared no conflict of interest.

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