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Stationary plasma configurations in a toroidal trap with D-shaped cross section in the two-fluid MHD approximation (Morozov–Soloviev equations)

V V Savelyev[1,2,3]

[1] Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya sq., 4, Moscow, 125047 Russia
[2] National Research Nuclear University MEPhI, Kashirskoe highway, 31, Moscow, 115409 Russia

[3] E-mail: ssvvvv@rambler.ru

Abstract. In 1974, A.I. Morozov and L.S. Soloviev derived a general set of hydrodynamic equations for two-component ideal plasma under the stationary flow conditions, as applied to the plasma acceleration problems. Under the conditions of axial symmetry, the authors managed to reduce this very complex system to a more simple form by introducing three functions of the flux (the magnetic field flux, electron flux and ion flux). This situation is reminiscent of the one that develops when obtaining the Grad–Shafranov equation: the problem of static plasma equilibrium is reduced to solving one second-order equation for the magnetic field flux function. The rest condition for the two-component plasma is usually interpreted as the condition of vanishing the mass-average plasma velocity. This condition coincides with the condition of ion immobility up to a value of the order of the electron to ion mass ratio. In this work, the Morozov–Soloviev equations will be used for the first time to study the stationary plasma configurations in the toroidal magnetic trap with the D-shaped cross section. The geometric parameters of the trap correspond to those of the two currently operating tokamaks: the JET and the JT60U.

1. Morozov–Soloviev equations (MS equations) for plasma at rest

The authors of [1, 2] proceed from the complete set of the stationary equations of two-fluid magnetic hydrodynamics for the ideal quasi-neutral ($n_e = n_i = n$) electron–ion plasma. For plasma at rest, using the conventional notations, it can be written as follows

$$\begin{gathered} div\, n\mathbf{V}_e = 0 \\ m_e n(\mathbf{V}_e, \nabla \mathbf{V}_e) = -\nabla p_e - en\,(\mathbf{E} + \frac{1}{c}[\mathbf{V}_e, \mathbf{B}]) \;,\; 0 = -\nabla p_i + en\,\mathbf{E}\,. \\ (\mathbf{V}_e, s_e) = 0 \;,\; p_e = p_e(n, s_e) \;,\; p_i = p_i(n) \\ rot\mathbf{B} = -\frac{4\pi}{c} en\mathbf{V}_e \;,\; div\mathbf{B} = 0 \;,\; \mathbf{E} = -\nabla\Phi \end{gathered} \tag{1}$$

Additionally, in Eqs. (1), the ions are assumed to be isentropic. Even in the axially symmetric case ($\partial / \partial \varphi = 0$), in cylindrical coordinates (r, φ, z), set of Eqs. (1) is very cumbersome and hard to solve. But it can be rewritten in the simpler form, using only two functions: the magnetic flux function $\Psi(r,z)$ and the total current function $J(r,z)$ (the level lines $\Psi(r,z) = const$ are the lines of the magnetic field in the (r, z) plane, and the level lines $J(r,z) = const$ are the electron current lines):

$$\begin{gathered} B_r = -\frac{1}{2\pi r}\frac{\partial \Psi}{\partial z} \;,\; B_\varphi = \frac{2J}{cr} \;,\; B_z = \frac{1}{2\pi r}\frac{\partial \Psi}{\partial r}, \\ j_r = -\frac{1}{2\pi r}\frac{\partial J}{\partial z} \;,\; j_\varphi = -\frac{c}{8\pi^2 r}\Delta^* \Psi \;,\; j_z = \frac{1}{2\pi r}\frac{\partial J}{\partial r} \;,\; \Delta^* \Psi = \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} \end{gathered} . \quad (2)$$

In terms of these two functions, the MS equations for the plasma at rest have the following form:

$$\begin{gathered} \Delta^* \Psi - \frac{8\pi^2 e}{cm_e} n \left[\frac{1}{2\pi}\frac{e}{c}(\Psi + \pi r^2 H_z) + K(J)\right] = 0 \\ \frac{m_e}{4\pi^2 e^2}\left[\frac{\partial}{\partial z}\left(\frac{1}{rn}\frac{\partial J}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{rn}\frac{\partial J}{\partial r}\right)\right] - rn\left(\frac{dF}{dJ} - m_e T_e \frac{ds_e}{dJ}\right) - \frac{j_\varphi}{e}\frac{dK}{dJ} - \frac{1}{\pi c^2}\frac{J}{r} = 0 \end{gathered} . \quad (3)$$

Since there is no dissipation, the total electron energy (the Bernoulli integral) is conserved along the current streamlines:

$$F(J) = \frac{m_e}{2}\mathbf{V}_e^2 - e\Phi + m_e W_e \equiv \frac{m_e}{2e^2}\frac{\mathbf{j}^2}{n^2} + m_i W_i(n) + m_e W_e \quad (4)$$

and also, due to the axial symmetry, there is the following integral of the electron angular momentum:

$$K(J) = rm_e V_{e,\varphi} - \frac{1}{2\pi}\frac{e}{c}(\Psi + \pi r^2 H_z) \equiv -\frac{m_e}{en} r j_\varphi - \frac{e}{2\pi c}(\Psi + \pi r^2 H_z). \quad (5)$$

The $s_e = s_e(J)$, $W_e(n,J)$, $W_i(n)$ functions are the entropy and enthalpy of electrons, and the enthalpy of ions, respectively, and $H_z = const$ is the external longitudinal magnetic field (it appears, if, in the equations of motion (1), the external field $\mathbf{H} = (0,0,H_z)$ is added). Equations (3), together with relations (4) and (5), form a closed set of equations. Thus, the solution of equations of equilibrium configurations (3) depends on three, generally speaking, arbitrary functions $s_e(J)$, $K(J)$, $F(J)$. In the equilibrium configuration, the electric field is equal to $\mathbf{E} = \nabla P_i / (en)$.

For ideal electron and ion gases, we can write the following expressions (we consider the polytropic constants γ for electrons and ions to be equal):

$$W_e(n,J) = \frac{\gamma}{(\gamma-1)m_e} G_e(J) n^{\gamma-1} \;,\; W_i(n) = \frac{\gamma}{(\gamma-1)m_i} G_i n^{\gamma-1} \;,\; G_i = const. \quad (6)$$

Then, Eq. (3) for the J function and the Bernoulli law (4) can be written in the more expanded form:

$$\begin{gathered} \frac{m_e}{4\pi^2 e^2}\left[\frac{\partial}{\partial z}\left(\frac{1}{rn}\frac{\partial J}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{rn}\frac{\partial J}{\partial r}\right)\right] - rn\left(\frac{dF}{dJ} - \frac{n^{\gamma-1}}{\gamma-1}\frac{dG_e}{dJ}\right) - \frac{j_\varphi}{e}\frac{dK}{dJ} - \frac{1}{\pi c^2}\frac{J}{r} = 0 \\ F(J) = \frac{m_e}{2e^2}\frac{\mathbf{j}^2}{n^2} + \frac{\gamma}{(\gamma-1)}(G_i + G_e(J))n^{\gamma-1} \end{gathered} . \quad (7)$$

From the Bernoulli law (7), we obtain the algebraic equation for finding the density n. If (for the sake of simplicity) we set $\gamma = 3$, then this equation will become biquadratic, and we will obtain the explicit expression for n:

$$n^2 = \frac{Q \pm \sqrt{Q^2 - \frac{3m_e}{4\pi^2 e^2 r^2}(G_i + G_e(J))(\nabla J)^2}}{3(G_i + G_e(J))} . \tag{8}$$
$$Q = F(J) - \frac{1}{2m_e r^2}\left(\frac{1}{2\pi}\frac{e}{c}\Psi + K + \frac{r^2}{2}\frac{e}{c}H_z\right)^2$$

It can be shown (see [3]) that, in the range of parameters corresponding to the "+" sign (the "dense" plasma), set of Eqs. (3) is elliptic, and, for the "–" sign (the "rarefied" plasma), it is hyperbolic. Here, we will consider only the elliptic case: the "+" sign in formula (8).

2. The boundary-value problem for the magnetic trap with the D-shaped cross section

The D-shaped cross section of the axisymmetric toroidal chamber is shown in figure 1 (the trap with the circular-shaped cross section was considered in [4]). It is constructed as follows. An isosceles triangle is constructed on the plane (r, z) with its vertex at the point $(r_{\max}, 0)$ and its base parallel to the z axis and passing through the point $(r_{\min}, 0)$. Then the three included angles of the triangle are smoothed by arcs of the circles, the centers of which lie on the bisectrices of the corresponding angles. The figure corresponds to the case when the large radius of the torus is approximately three times the small radius.

The following boundary conditions are set on the boundary of the region obtained in this way:

$$J = J_0 , \quad \Psi = \Psi_0 , \tag{9}$$

where J_0, Ψ_0 are the constants. The J_0 value is equal to the current in the external toroidal winding. The Ψ_0 value is equal to the magnetic field flux through the central hole of the torus.

For the sake of convenience of numerical solving, we introduce new units of measurement using the given and known physical parameters included in the statement of the problem. The unit of length L_0 will be the "small radius" of the torus. The magnetic field will be measured in units of $B_0 = 2J_0 / cL_0$. The units of density and temperature are constructed from the known values of the $F(J) = F_0 f(J/J_0)$, $G_e(J) = G_{e0} g(J/J_0)$, $K(J) = K_0 k(J/J_0)$ functions, namely:

$$T_{e0} = F_0 / k_B , \quad n_0^2 = \frac{k_B T_{e0}}{G_{e0}} , \quad T_{i0} = \frac{n_0^2 G_{i0}}{k_B},$$

where k_B is the Boltzmann constant.

In new units, relations (4) and (5) can be written as follows

$$f(J) = \frac{\xi^2}{\beta}\frac{\mathbf{j}^2}{n^2} + \frac{3}{2}(\eta + g(J))n^2 , \tag{10}$$

$$k(J) = -\xi^2 \frac{r j_\varphi}{n} - \Psi - \alpha k(J) - \frac{r^2}{2} h_z . \tag{11}$$

And finally, the basic equations (3) in the new units ($\Psi = rA$) take the following form:

$$\xi^2\left(\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial rA}{\partial r}\right)\right) - \frac{n}{r}\left(rA + \alpha k(J) + \frac{r^2}{2}h_z\right) = 0$$
$$\xi^2\left[\frac{\partial}{\partial z}\left(\frac{1}{rn}\frac{\partial J}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{rn}\frac{\partial J}{\partial r}\right)\right] - \frac{\beta}{2} rn\left(\frac{df}{dJ} - \frac{1}{2}n^2\frac{dg(J)}{dJ}\right) - \alpha j_\varphi \frac{dk}{dJ} - \frac{J}{r} = 0 . \tag{12}$$

Formula for density is as follows:

$$n^2 = \frac{q + [\, q^2 - 6\frac{\xi^2}{\beta r^2}(\,\eta + g(J)\,)(\nabla J)^2\,]^{1/2}}{3(\,\eta + g(J)\,)} \quad . \tag{13}$$
$$q = f(J) - \frac{1}{\xi^2 \beta r^2}(\Psi + \alpha k(J) + \frac{r^2}{2} h_z)^2$$

and also we have the following expressions for the dimensionless parameters:

$$\xi^2 = \frac{c^2 m_e}{4\pi e^2 L_0^2 n_0} \;,\quad \beta = \frac{8\pi k_B n_0 T_{e0}}{B_0^2} \;,\quad \eta = \frac{T_{i0}}{T_{e0}}$$
$$\alpha = \frac{c^2 K_0}{2eL_0 J_0} = \frac{c}{e}\frac{K_0}{B_0 L_0^2} \;,\quad \delta = \frac{B_{p0}}{B_0} \;,\quad h_z = \frac{H_z}{B_0} \quad . \tag{14}$$

In new units, conditions (9) take the following form:

$$J = 1 \quad , \quad \Psi = \frac{B_{p0}}{B_0} = \delta \quad , \quad B_{p0} = \frac{\Psi_0}{2\pi L_0^2} \, . \tag{15}$$

We also have three arbitrary functions $g(J)\,,f(J)\,,k(J)$ with the following normalization: $g(1) = f(1) = k(1) = 1$. In [3], it was considered in detail, how Eqs. (12) relate to the Grad-Shafranov equation [5]. Among the dimensionless parameters (14), the parameter ξ^2 is of particular importance. It is shown that the Grad-Shafranov equation can be obtained from Eqs. (12) in a limit of $\xi^2 \to 0$. Since the parameter ξ^2 is included singularly in the set of Eqs. (12) (it stands at the highest derivatives), the Grad-Shafranov equations are not only the limiting, but also the degenerate case of the Morozov-Soloviev equations. Some examples can be given. If, when solving the equation, the characteristic size of inhomogeneity is ~1 cm, then, at densities of n ~10^{12} and ~10^{11} cm^{-3}, the ξ^2 values will be ~0.3 and ~3, respectively. It is also important to note that the small parameter $\mu = m_e / m_i$ does not appear in the problem at all.

The second-order operator in the left-hand side of the second of Eqs. (12) is the azimuthal component of *rot* $\mathbf{V}_e$, up to a constant factor, and therefore, the electron motion in the plane (r, z) will always be the vortex motion.

It is easy to obtain the necessary condition for the existence of a solution of the boundary-value problem for Eqs. (12). In order to obtain the positive density in expressions (13), it is necessary that the q value should be always positive. But it is known, and it can be easily calculated at any point on the region boundary, for example, at the point $(r_{\min}, 0)$. Then we get the necessary condition for the existence of a solution of the boundary-value problem for Eqs. (12) that can be easily verified:

$$\xi^2 \beta > (\alpha + \delta + \frac{r_{\min}^2}{2} h_z)^2 / r_{\min}^2 \, . \tag{16}$$

In currently operating traps, the β and ξ^2 parameters take simultaneously rather small values. In this case, it can be seen from condition (16) that, other factors being equal, the lesser is the $r_{\min}$ radius, i.e. the smaller is the distance from the camera to the z axis, the harder it is to satisfy this condition. For $h_z = 0$ and $\alpha = -\delta$, condition (16) is always satisfied.

3. Results of numerical research

The boundary value problem (12), (15) is a very complicated problem involving the nonlinear elliptic set of the second-order equations. It involves a lot of numerical parameters (14) and three parameter functions $g(J)$, $f(J)$, $k(J)$.

The parameters of the operating toroidal magnetic traps (geometric dimensions, fields and currents) may be greatly different. There are a lot of publications presenting the results of the theoretical and experimental research performed at these facilities [6, 7]. The results of this work were obtained not for the particular facility with its specific parameters, although the main geometric parameters used correspond to two currently operating tokamaks: the JET and the JT60U [7]. The main goal of calculations performed in the new geometry was (similarly to [4]) to make sure that Eqs. (12) have solutions, which provide plasma confinement, i.e. the solutions describing plasma, in which the plasma pressure (or its density) is maximal in the central region of the chamber, and it is considerably lower in all near-boundary regions of plasma.

The following forms of parameter functions were used in the calculations:

$$f(J)=1+q_f(J-1)^2\,,\quad g(J)=1+q_g(J-1)^2\,,\quad k(J)=1+q_k(J-1)^2\,. \tag{17}$$

Here, q_f, q_g, q_k are the constant numbers. For real plasma systems, the exact form of these three functions is not known.

Next, in accordance with the main goal of the work, we present the results of only one calculation. The level lines of the Ψ, j_φ, p_e, $\mathbf{E}$ functions are presented in figure 1. They were obtained at the following parameters:

$$\xi^2=0.05\,,\beta=10\,,\alpha=0.9\,,\delta=0.025\,,\ \eta=0.1\,,h_z=0.2\,,q_f=0.5\,,q_g=0.3\,,q_k=0\,. \tag{18}$$

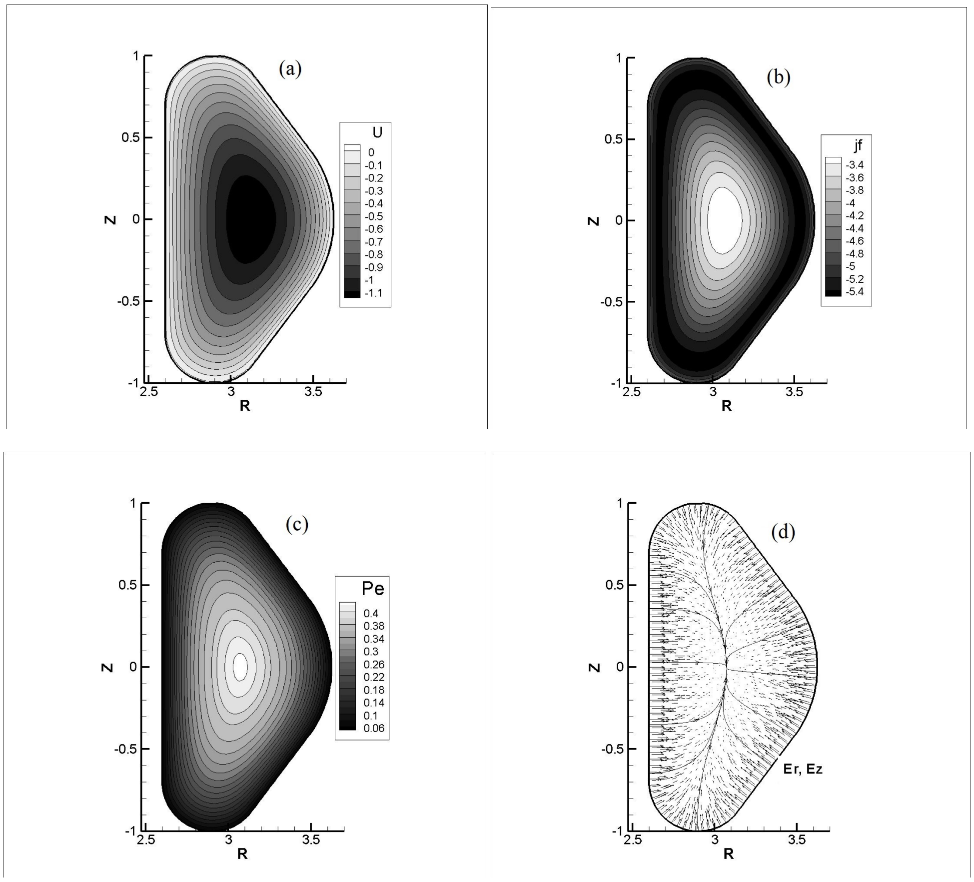

Figure 1. Level lines of the (a) Ψ, (b) j_φ, and (c) p_e functions; (d) field distribution (E_r, E_z).

Since the ξ^2 parameter given by expressions (18) is rather small, therefore, the level lines of the Ψ, J, p_e functions are similar to each other. At large parameters ξ^2, the level lines of these three functions will be considerably different. The pressure is maximal in the center of the chamber and decreases to its borders. On the contrary, the azimuthal current has the maximum at the periphery of the compression region. Ions are confined by only the electric field.

When solving the problem numerically, we used the method of finite elements of the second order applied to the triangular grid.

4. Conclusions

In this work, we presented the derivation of the Morozov–Soloviev equations for the plasma at rest. The possibilities provided by equations, as applied to the theory of equilibrium configurations in plasma traps, are illustrated by the first results of numerical studies of the boundary-value problem for the traps with the D-shaped cross section.

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