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Comparison of numerically-analytical and finite-element solutions of the Lamé problem for nonlinear-elastic cylinder under large strains

K M Zingerman^{1,2}, A V Vershinin³ and V A Levin³

¹Tver State University, 33 Zhelyabov Street, Tver 170100, Russian Federation

²National Research Nuclear University MEPhI 115409, Russian Federation, Moscow, Kashirskoe shosse, 31

³Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, 119991, Moscow, GSP-1, 1 Leninskiye Gory, Main Building, Russian Federation

E-mail: zingerman@rambler.ru

Abstract. The results of a comparison of the numerical-analytical solution of the Lamé problem for a cylinder made of nonlinear elastic material with the solution of the same problem by the finite element method are presented. Calculations are made for the compressible Mooney-Rivlin material and the Murnaghan material. The numerical-analytical solution is based on the use of the semi-inverse method, which is briefly described in relation to the class of problems under consideration; the problem reduces to solving the Cauchy problem for an ordinary differential equation. A finite-element solution was found in a three-dimensional formulation using the Fidesys engineering strength analysis system using 8-node and 20-node elements. The influence of the type of finite elements, material parameters and the magnitude of the applied load on the computational error is investigated.

1. Introduction

Nonlinear elasticity problems for compressible materials under large strains, as a rule, do not have an analytical solution (except for certain classes of materials, such as semilinear material or a special case of the Blatz-Ko material [1-10], and some classes of nonstationary problems such as ones on the propagation of waves [11] or ones on phase transitions [12, 13]). In this regard, it is of interest to compare solutions obtained using various approaches. In [14], a method of numerical-analytical solution of the Lamé problem for a hollow cylinder made of a nonlinear-elastic compressible material was proposed for large strains and a comparison with a finite-element solution of the same problem was performed. The present paper generalizes this method to the case of the Murnaghan material, and also investigates the dependence of the computational error of the finite-element solution on the element type (8-node and 20-node three-dimensional elements are used). For the compressible Mooney-Rivlin material, the influence of the material compressibility on the calculation results has been studied.

2. Problem statement

The problem is formulated in the coordinates of the initial (undeformed) state. The equilibrium equation has the form



$$\overset{0}{\nabla} \cdot \mathbf{P} = 0 \quad (1)$$

Here \mathbf{P} is the first Piola-Kirchhoff stress tensor. This tensor is expressed in terms of the true stress tensor $\boldsymbol{\sigma}$ as follows:

$$\mathbf{P} = J (\boldsymbol{\Psi}^{-1})^T \cdot \boldsymbol{\sigma} \quad (2)$$

$\boldsymbol{\Psi} = \overset{0}{\nabla} \mathbf{R}$ – a strain gradient; \mathbf{R} is the radius vector of the particle under strain; $J = \det \boldsymbol{\Psi}$ – the relative volume change.

The boundary conditions on the inner lateral cylinder surface are written in the form

$$\mathbf{n} \cdot \mathbf{P} = -p \mathbf{j} \mathbf{n} \cdot (\boldsymbol{\Psi}^{-1})^T \quad (3)$$

Here p is the pressure defined on the boundary; \mathbf{n} is the unit normal vector to the boundary in the unstrained state. The external lateral cylinder surface is load-free, and on the bases of the cylinder there are no displacements in the direction of its axis.

The constitutive equations generally have the form

$$\mathbf{P} = \frac{dW}{d\boldsymbol{\Psi}} \quad (4)$$

Here, W is an elastic potential. For the compressible Mooney-Rivlin material, the potential is given in the form

$$W = C_1 \bar{I}_1 + C_2 \bar{I}_2 + D(J-1)^2 \quad (5)$$

Here C_1, C_2, D are constants of the material; $\bar{I}_1 = J^{-2/3} I_1$; $\bar{I}_2 = J^{-4/3} I_2$; I_1, I_2 are invariants of the Green stress tensor measure $\mathbf{G} = \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^T$. The constant D characterizes the bulk stiffness of the material.

For the Murnaghan material, the elastic potential has the form

$$W = \frac{1}{2} \lambda (E_1)^2 + \mu E_2 + C_3 (E_1)^3 + C_4 E_1 E_2 + C_5 E_3 \quad (6)$$

Here $E_k = \left(\overset{0}{\mathbf{E}} \right)^k \cdot \mathbf{I}$ ($k=1,2,3$) are the Green stress tensor invariants

$$\overset{0}{\mathbf{E}} = \frac{1}{2} (\mathbf{G} - \mathbf{I}) = \frac{1}{2} (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^T - \mathbf{I}) \quad (7)$$

\mathbf{I} is the identity tensor; $\lambda, \mu, C_3, C_4, C_5$ are the material constants (λ, μ are the Lamé constants).

3. Problem-solving methods

The numerical-analytical solution has been obtained using the semi-inverse method [14]. Taking into account the symmetry of the problem, the general form of the solution is taken in the form

$$R = R(r), \quad Z = z, \quad \Phi = \varphi \quad (8)$$

Here (r, z, φ) are the cylindrical coordinates of the particle in the unstrained state; (R, Z, Φ) are the cylindrical coordinates of the particle in the final state; $R(r)$ is a function to be determined.

The diagonal components of the strain gradient, taking into account (8), can be expressed in terms of the function $R(r)$ as follows:

$$\Psi_{rr} = R'(r), \quad \Psi_{\varphi\varphi} = R(r)/r, \quad \Psi_{zz} = 1 \quad (9)$$

The remaining components of the strain gradient are equal to zero.

Substitution of expressions (9) into the constitutive equations (4) allows us to express the components of the first Piola-Kirchhoff stress tensor \mathbf{P} through the function $R(r)$. Further, the components of the tensor \mathbf{P} are substituted into the equilibrium equation (1). As a result, the problem solution reduces to solving an ordinary nonlinear second-order differential equation with respect to an unknown function $R(r)$. The boundary conditions for the equation are written based on the boundary condition (3), and can be represented in the form

$$P_{rr}(a) = 0, P_{rr}(b) = -p\Psi_{\varphi\varphi}(b) \quad (10)$$

Here a and b are the outer and inner radii of the cylinder, respectively.

Along with the boundary-value problem with boundary conditions (10), we can consider the Cauchy problem for the same equation with given values $R(a)$ and $R'(a)$. The value $R'(a)$ is determined from the condition of no load on the outer boundary. For this purpose a nonlinear algebraic equation $P_{rr}(a) = 0$ is solved with respect to $\Psi_{rr}(a)$ with the given value $\Psi_{\varphi\varphi}(a) = R(a)/a$. The found value $\Psi_{rr}(a)$, in accordance with (9), equals to $R'(a)$.

Solving the Cauchy problem, we can find, in particular, the pressure p at the inner boundary (at $r = b$). Solving this problem for a set of given values $R(a)$, it is possible to compile a table of the dependence of the pressure p at the inner boundary and other stress-strain state parameters on the outer radius of the cylinder after strain. When such a table is constructed, by means of interpolation, we can approximately find the stress-strain state parameters for a given pressure p .

To find the finite-element solution, the Fidesys engineering strength analysis system [16-18] (version 1.7) was used. The calculations have been performed in a three-dimensional formulation for two types of hexahedral elements: 8-node and 20-node.

4. Calculation results

All calculations were carried out for the cylinder with a ratio of the inner radius to the outer one $b/a = 0.6$ with height $h = 2a$. Calculations have been performed for a quarter of the cylinder. The finite-element mesh is shown in Fig. 1. The mesh contains 1092 elements and 8736 nodes (for 8-node elements).

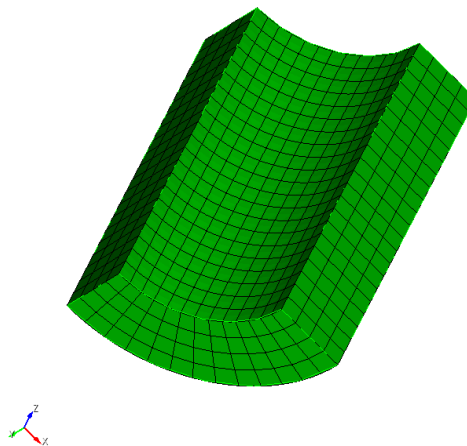


Fig. 1. Finite-element mesh

4.1. Results for the Mooney-Rivlin material

Calculations for the potential (5) are performed at $C_2 = 0$ for four values of bulk stiffness of the material: $D/C_1 = 2, 4, 8$ and 16 . Calculations are made for three values of internal pressure: $p/C_1 = 0.25, 0.5$ and 0.7 . The results are shown in Tables 1–4. Each of the tables contains results for a specific bulk stiffness values D . The tables give the values of the true circumferential stress $\sigma_{\varphi\varphi}$ at

the inner boundary of the cylinder for the values of the internal pressure indicated in the first column. The second column contains the results for the numerical-analytical solution; the third column shows the results for the finite-element solution using 8-node elements; the fourth column – for the finite-element solution using 20-node elements.

Table 1. The true circumferential stress $\sigma_{\varphi\varphi}$ at the internal boundary of the cylinder for different values of internal pressure p with $D/C_1 = 2$. Mooney-Rivlin material. All values are assigned to C_1 .

p/C_1	Numerical-analytical	FEM, 8-node element	FEM, 20-node element
0.25	0.632	0.644	0.633
0.50	1.651	1.662	1.653
0.70	3.681	3.638	3.681

Table 2. The same as in Table 1, where $D/C_1 = 4$.

p/C_1	Numerical-analytical	FEM, 8-node element	FEM, 20-node element
0.25	0.632	0.675	0.637
0.50	1.633	1.706	1.641
0.70	3.300	3.358	3.306

Table 3. The same as in Table 1, where $D/C_1 = 8$.

p/C_1	Numerical-analytical	FEM, 8-node element	FEM, 20-node element
0.25	0.632	0.738	0.645
0.50	1.625	1.825	1.645
0.70	3.176	3.414	3.194

Table 4. The same as in Table 1, where $D/C_1 = 16$.

p/C_1	Numerical-analytical	FEM, 8-node element	FEM, 20-node element
0.25	0.632	0.862	0.660
0.50	1.622	2.073	1.666
0.70	3.125	3.719	3.169

The tables show that the computational error (the difference module between the numerical-analytical and the finite-element solutions) for the solution using 8-node elements is much larger than for the solution using 20-node elements. The computational error also goes up with the increasing volumetric stiffness D . The growth of the computational error with increasing bulk stiffness can be explained by the known effect of the elements "locking" [19-21]. This effect for the 20-node elements

apparently manifests itself in the insignificant dependence of the principal stresses inside the cylinder on the angular coordinate for sufficiently large values of D , for example, at $D/C_1 = 16$ (Fig. 2).

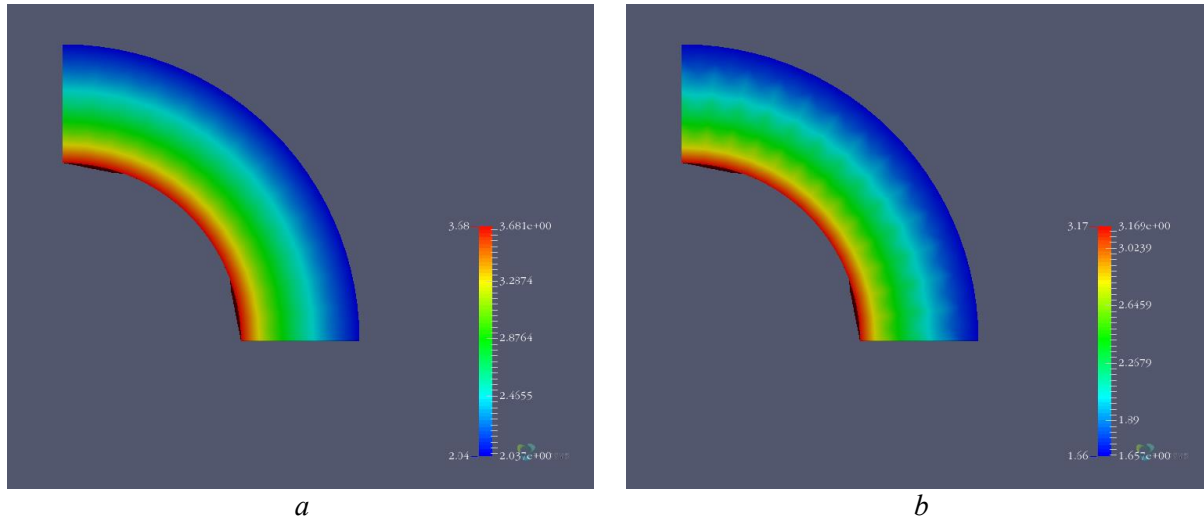


Fig. 2. Distribution of the maximum principal true stress in the cylinder made of the Mooney-Rivlin material at $p/C_1 = 0.25$. $a - D/C_1 = 2$; $b - D/C_1 = 16$.

Maximum (according to Tables 1–4) relative computational error of the finite-element solution is achieved at $p/C_1 = 0.25$, $D/C_1 = 16$ and is 36% for the 8-node elements and 4% for the 20-node elements. Taking it into consideration, it can be concluded that the numerical-analytical solution for the compressible Mooney-Rivlin material agrees with the finite-element solution when using 20-node hexahedral elements.

4.2. Results for the Murnaghan material

Calculations for the potential (6) are performed for $\lambda/\mu = 2$, $C_3 = C_4 = C_5 = 0$ (i.e. for the physically linear material). Calculations are carried out for three values of internal pressure: $p/\mu = 0.1, 0.2$ and 0.3 . The results are shown in Table 5, having the same structure as the previous tables. From the table it can be seen that, as for the Mooney-Rivlin material, the 20-node elements give more accurate results than the 8-node elements.

Table 5. The true circumferential stress $\sigma_{\varphi\varphi}$ at the internal boundary of the cylinder for different values of internal pressure p . The Murnaghan material. All values are assigned to μ .

p/μ	Numerical-analytical	FEM, 8-node element	FEM, 20-node element
0.1	0.263	0.274	0.266
0.2	0.680	0.695	0.687
0.3	1.585	1.437	1.551

Maximum (according to Table 5) relative computational error of the finite-element solution is achieved at $p/\mu = 0.3$, and is 9% for the 8-node elements and 2% for the 20-node elements. With this

in mind, we can conclude that the numerical-analytical solution for the Murnaghan material agrees with the finite-element solution when using 20-node hexahedral elements.

5. Conclusion

Thus, the results obtained make it possible to conclude that the numerical-analytical and finite-element solutions of the Lamé problem are well-suited under large strains in the investigated range of loads (pressure) and material parameters for the Mooney-Rivlin (compressible) and Murnaghan materials using 20-node hexahedral elements. For the weakly compressible Mooney-Rivlin material, the computational error of the finite-element solution grows with increasing bulk stiffness.

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