Fortschritte der Physik
Progress of Physics
www.fp-journal.org

Quantum Fluctuations and New Instantons I: Linear Unbounded Potential

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We consider the decay of a false vacuum in circumstances where the methods suggested by Coleman run into difficulties. We find that in these cases quantum fluctuations play a crucial role. Namely, they naturally induce both ultraviolet and infrared cutoff scales, determined by the parameters of the classical solution, beyond which this solution cannot be trusted anymore. This leads to the appearance of a broad class of new $O(4)$ invariant instantons, which would have been singular in the absence of an ultraviolet cutoff. We apply our results to a case where the potential is unbounded from below in a linear way and in particular show how the problem of small instantons is resolved by taking into account the inevitable quantum fluctuations.

1. Introduction

The study of the details of the fate of a false vacuum plays a key role in understanding the properties of a variety of systems. It extends from the understanding the characteristics of various phase transitions in condensed matter and particle physics all the way to quantifying the human angst that our universe itself may actually be in a state of a false vacuum. The gross features of the decay mechanism, the formation of bubbles and the quantum under the barrier tunneling involve both the classical and quantum aspects of the problem. The description of tunneling in the non-relativistic quantum mechanics for a particle moving in many dimensions was considered in [1]. In quantum field theory the problem of a false vacuum instability was first addressed in [2], assuming that the thickness of the wall is small compared to the size of the bubble. Coleman[3] has generalized the results of [2] and suggested how to encapsulate and calculate the non-perturbative quantum part of the process by identifying Euclidean classical configurations which give the leading contribution to the probability of the false vacuum decay. In this paper we study cases where the method suggested by Coleman, as is, needs to be reexamined and modified. In particular, we investigate how the theory must be modified by taking into account the inevitable quantum fluctuations of the scalar field.

We start by a short review of the Coleman theory for the scalar field potential $V(\varphi)$, which has a shape shown in **Figure 1** (for details see, for example [4, 5]). The false vacuum configuration $\varphi(\mathbf{x}) = \varphi_0$ is unstable and decays via bubble nucleation. The field inside the emerged expanding bubble either tends to its value in the true minimum or $\varphi \to -\infty$ for the unbounded potential.

The scalar field $\varphi(\mathbf{x}, t)$ is a system with infinitely many degrees of freedom with spatial coordinates $\mathbf{x} = (x^1, x^2, x^3)$ enumerating them, that is, $\varphi(\mathbf{x}, t) = \varphi_{\mathbf{x}}(t)$. The scalar field action can be written as

$$S = \int (\mathcal{K} - \mathcal{V})\, dt, \tag{1}$$

where

$$\mathcal{K} = \frac{1}{2} \int \left(\frac{\partial \varphi}{\partial t}\right)^2 d^3x, \tag{2}$$

and

$$\mathcal{V} = \int \left(\frac{1}{2}\left(\frac{\partial \varphi}{\partial x^i}\right)^2 + V(\varphi)\right) d^3x \tag{3}$$

are, correspondingly, the kinetic and interaction energies of the field configuration $\varphi(\mathbf{x})$ and its first time derivatives. Note that the functional $\mathcal{V}(\varphi(\mathbf{x}))$ (*but not* $V(\varphi)$) plays the role of the potential energy when we are considering the sub-barrier tunneling in quantum field theory. For the scalar field potential $V(\varphi)$, shown in Figure 1, the field configuration $\varphi(\mathbf{x}) = \varphi_0$ has zero potential energy, $\mathcal{V}(\varphi_0) = 0$, while the other homogeneous static

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DOI: 10.1002/prop.202000100

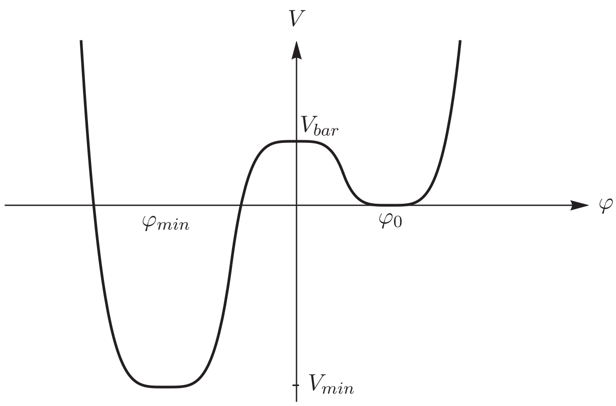

Figure 1.

solution $\varphi(\mathbf{x}) = \varphi_{\min}$ has a lower energy $\mathcal{V}(\varphi_{\min}) = V_{\min}\times volume < 0$. Therefore the local minimum at φ_0 is unstable (false vacuum) and decays via sub-barrier tunneling as a result of which the bubbles of the true vacuum are formed. To calculate the decay rate Coleman has assumed that in the semiclassical approximation the dominant contribution to the tunneling rate comes from the instanton - Euclidean solution of the scalar field equation, which matches the metastable vacuum $\varphi(\mathbf{x}) = \varphi_0$ to some classically allowed configuration $\varphi(\mathbf{x})$ with $\mathcal{V}(\varphi(\mathbf{x})) = 0$, describing the emerging bubble in Minkowski space. On symmetry grounds it was shown[6] that the minimal action has a $O(4)$ -invariant solution of the scalar field equation

$$\frac{\partial^2\varphi}{\partial\tau^2} + \Delta\varphi - \frac{dV}{d\varphi} = 0, \tag{4}$$

for which φ depends only on $\varrho = \sqrt{\tau^2 + \mathbf{x}^2}$ and where $\tau = it$ is the Euclidean time. This reduces to the ordinary differential equation

$$\ddot{\varphi}(\varrho) + \frac{3}{\varrho}\,\dot{\varphi}(\varrho) - \frac{dV}{d\varphi} = 0, \tag{5}$$

where dot denotes the derivative with respect to ϱ. If we assume that the field was initially in the false vacuum state one of the boundary conditions for (5) is

$$\varphi(\varrho \to \infty) = \varphi_0. \tag{6}$$

As a second condition Coleman has suggested to use

$$\dot{\varphi}(\varrho = 0) = 0 \tag{7}$$

to avoid a singularity in the center of the bubble. Equation (5) with *these boundary conditions* has an unambiguous solution $\varphi(\varrho)$ called the Coleman instanton. The action for this instanton is given by

$$S_I = 2\pi^2 \int_0^{+\infty} d\varrho\, \varrho^3 \left(\frac{1}{2}\,\dot{\varphi}^2 + V(\varphi)\right) \tag{8}$$

and the false vacuum decay rate *per unit time per unit volume* can be estimated as

$$\Gamma \simeq \varrho_0^{-4} \exp\left(-S_I\right), \tag{9}$$

where ϱ_0 is the size of the bubble. The pre-exponential factor is based on dimensional grounds. Calculating the potential energy (3) at the moment of Euclidean time τ, we infer that $\mathcal{V}(\tau) > 0$ for $0 < \tau < \infty$. Since the total energy is normalized to zero this means that in this range of τ the instanton describes the sub-barrier tunneling in the Euclidean time. As $\tau \to \infty$, $\mathcal{V} \to 0$, corresponding to the false vacuum state. The potential energy also vanishes at $\tau = 0$ and hence at this instant the bubble emerges from under the barrier in Minkowski space. To prove that $\mathcal{V}(\tau = 0) = 0$ we first note that for $\varphi(\mathbf{x}, \tau) = \varphi(\varrho)$ the expression (3), calculated at $\tau = 0$, reduces to

$$\mathcal{V}(\tau = 0) = 4\pi \int_0^{+\infty} d\varrho\, \varrho^2 \left(\frac{1}{2}\,\dot{\varphi}^2 + V(\varphi)\right). \tag{10}$$

Next we find that the first integral of (5) can be written as

$$\frac{1}{2}\dot{\varphi}^2 - V = \int_\varrho^\infty \frac{3}{\tilde{\varrho}}\left(\frac{d\varphi}{d\tilde{\varrho}}\right)^2 d\tilde{\varrho}, \tag{11}$$

where the boundary condition (6) has been taken into account. Finally integrating this equation one gets[1]

$$\int_0^\infty \left(\frac{1}{2}\,\dot{\varphi}^2 - V\right)\varrho^2 d\varrho = \int_0^\infty d\left(\varrho^3\right)\left(\int_\varrho^\infty \frac{1}{\tilde{\varrho}}\left(\frac{d\varphi}{d\tilde{\varrho}}\right)^2 d\tilde{\varrho}\right) = \int_0^\infty \dot{\varphi}^2\varrho^2 d\varrho \tag{12}$$

and hence

$$\int_0^\infty \left(\frac{1}{2}\,\dot{\varphi}^2 + V\right)\varrho^2 d\varrho = 0 \tag{13}$$

implying that $\mathcal{V}(\tau = 0)$ (10) really vanishes.

There is a class of potentials for which the results obtained using the Coleman boundary conditions can either lead to a questionable outcome or cannot be applied at all, namely,

a) for the very steep unbounded potentials (see Figure 3) Coleman's instantons lead to nearly instantaneous instability of the false vacuum, the so called zero size instanton problem,
b) for some unbounded potentials, when the false vacuum must be unstable, the Coleman instanton does not exist.

We will show that both problems have common origin and can be resolved by taking into account inevitable quantum fluctuations which induce the ultraviolet cutoff scale. In that situation the remedy suggested by Coleman to avoid a singularity at the origin needs to be and is replaced by an ultraviolet cutoff scale induced by those very quantum fluctuations. In turn, this allows us to abandon the very restrictive boundary condition (7) and obtain a whole class of new instantons which would be singular in the absence of this quantum ultraviolet cutoff.

[1] The second equality in (12) is the result of integration by parts taking into account (6) and (7) as well as requiring that $\varphi(\varrho)$ decays faster than $\varrho^{-\frac{1}{2}}$ at $\varrho \to \infty$.

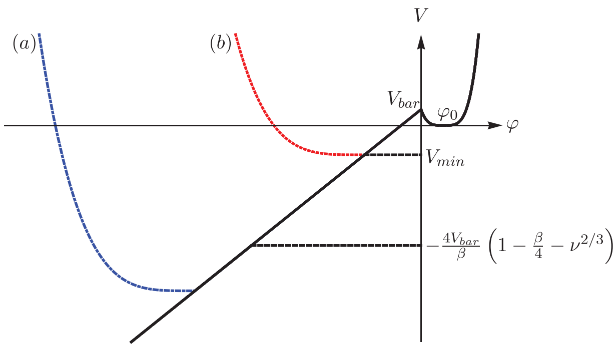

Figure 2.

In this paper we will consider an unbounded linear potential and clarify the role of quantum fluctuations in resolving the small instanton problem. The case of a quartic unbounded potential for which the Coleman instanton does not exist is analyzed in the companion paper.[7]

2. The Model

To liberate ourselves from the specificities of approximations, such as the thin wall one, we consider a classically exactly solvable potential. While its precise form may not naturally serve as a low energy description, it does share enough properties of the system we wish to study that we consider it well worthwhile paying the price. The potential is:

$$V(\varphi)=\begin{cases}\lambda_-\varphi_0^3\varphi+\frac{\lambda_+}{4}\varphi_0^4 & \text{for } \varphi<0,\\ \frac{\lambda_+}{4}\left(\varphi-\varphi_0\right)^4 & \text{for } \varphi>0.\end{cases} \tag{14}$$

It has a local minimum, corresponding to the false vacuum at φ_0. To avoid the problem with one loop quantum corrections, we will assume that the dimensionless coupling constant λ_+ is always much smaller than unity. At $\varphi=0$ the φ^4–potential is matched to a linear unbounded potential for which the dimensionless constant λ_- can be taken to be large.

The results obtained in this paper can also be applied to estimate the probability of tunneling for some potentials with a second true minimum at some negative $\varphi_{\min}$, which for $\varphi<0$ can well be approximated by the linear potential (14) nearly up to $\varphi_{\min}$ (see dotted line in **Figure 2**).

3. The Coleman Instanton

First we find the explicit exact solution for the Coleman instanton and show how the problem mentioned in the introduction arises in this simple particular case. For positive φ equation (5) takes the following form

$$\ddot{\varphi}+\frac{3}{\varrho}\dot{\varphi}-\lambda_+\left(\varphi-\varphi_0\right)^3=0 \tag{15}$$

and its solution with the boundary condition (6) is

$$\varphi(\varrho)=\varphi_0\frac{\varrho^2-\varrho_0^2}{\varrho^2-\varrho_0^2/(1+\delta)}, \tag{16}$$

where

$$\delta=\frac{4\left(1+\sqrt{1+\lambda_+\varphi_0^2\varrho_0^2/2}\right)}{\lambda_+\varphi_0^2\varrho_0^2} \tag{17}$$

and the constant of integration ϱ_0 (the size of the bubble) remains yet to be fixed with the help of the second boundary condition (7). The solution (16) is valid only for $\varrho_0<\varrho<\infty$. At $\varrho=\varrho_0$ the field φ vanishes and then becomes negative. For $\varphi<0$ the potential is linear and equation (5) simplifies to

$$\ddot{\varphi}+\frac{3}{\varrho}\dot{\varphi}-\lambda_-\varphi_0^3=0. \tag{18}$$

The solution of this equation, which vanishes at ϱ_0 and satisfies (7), is

$$\varphi(\varrho)=\frac{\lambda_-\varphi_0^3}{8}\left(\varrho^2-\varrho_0^2\right). \tag{19}$$

The derivative of the field φ at $\varrho=\varrho_0$ must be continuous. This allows us to express the size of the bubble in terms of the parameters λ_+, λ_- and φ_0. Equating the derivatives of solutions (16) and (19) at ϱ_0 we obtain the following equation

$$1+\frac{1}{\delta}=\frac{\lambda_-\varphi_0^2}{8}\varrho_0^2 \tag{20}$$

with δ given in (17). Solving this equation for ϱ_0^2 one gets

$$\varrho_0^2=\frac{8}{\varphi_0^2\lambda_-(1-\beta)}, \tag{21}$$

where we have introduced the parameter

$$\beta=\frac{\lambda_+}{\lambda_++\lambda_-} \tag{22}$$

instead of λ_+.

As one can see from (16) for $\delta\ll 1$ the bubble has a thin wall. Substituting ϱ_0^2 from (21) into equation (20) we find

$$\delta=\frac{1-\beta}{\beta} \tag{23}$$

and therefore the thin-wall approximation is valid only if $1-\beta\ll 1$, that is, for a rather flat potential ($\lambda_-\ll\lambda_+$) at negative φ. As it follows from (19) the value of the scalar field in the center of the bubble is equal to

$$\varphi(\varrho=0)=-\frac{\varphi_0}{1-\beta}, \tag{24}$$

which corresponds to

$$V(\varphi(\varrho=0))=-\frac{\lambda_-\varphi_0^4}{1-\beta}\left(1-\frac{\beta}{4}\right)=-\frac{4V_{bar}}{\beta}\left(1-\frac{\beta}{4}\right), \tag{25}$$

where

$$V_{bar}=\lambda_+\varphi_0^4/4 \tag{26}$$

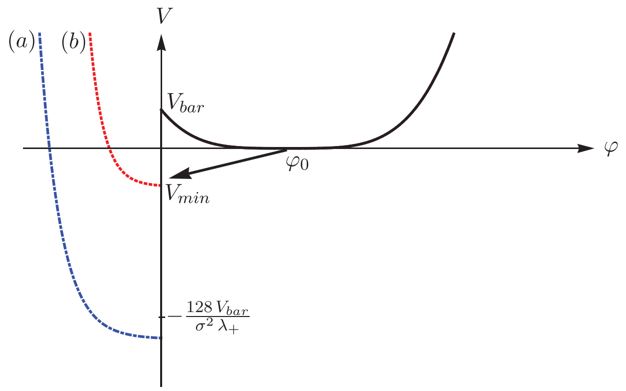

Figure 3.

is the height of the barrier. The instanton action can be calculated and is equal to

$$S_I = \frac{8\,\pi^2}{3\,\lambda_-(1-\beta)^3}\,(2-\beta)\,(2-2\beta+\beta^2). \tag{27}$$

We note that if one would decide to consider $\lambda_- \to \infty$, $\beta \to 0$ the value of the scalar field in the center of the bubble is equal to $\varphi(\varrho = 0) \to -\varphi_0$ and $V(\varphi(\varrho = 0)) \to -\infty$.

Instantaneous vacuum decay via small bubbles. For a very steep unbounded potential shown in **Figure 3** and obtained by taking the limit $\lambda_- \to \infty$, the bubble size given in (21) shrinks to zero as $\varrho_0 \propto \lambda_-^{-1/2}$ and the action (27) vanishes. The false vacuum decay rate (9) becomes infinite irrespective how high is the potential barrier V_{bar}. This instantaneous false vacuum decay happens via infinitely small bubbles, which according to (25) emerge with infinitely large negative potential in the center. Such a conclusion does not sound physically acceptable. Moreover, even for finite but large enough λ_- the problem of small instantons still remains because they lead to unexpectedly efficient decay with the rate practically independent on the shape of the potential at positive φ. The situation starts to look even more strange if one assumes that the potential in Figure 3 gets a second rather sharp minimum (see dotted line). Then, irrespective of the depth of this second minimum, it looks like the Coleman instanton ceases to exist and, thus, nearly irrelevant modification of the potential abruptly changes the decay rate from infinity to zero.

Below we will show how these problems are resolved if we take into account quantum fluctuations.

4. Quantum Fluctuations

The instanton is a *classical* solution and it is reliable only if the contributions both the field and its derivative exceed the level of the minimal quantum fluctuations. The bubble emerges in Minkowski space at $\tau = 0$. At this instant the distance to the center of the bubble is equal to ϱ and the "typical" amplitude of the quantum fluctuations in corresponding scales is about (see for example [8]):

$$\left|\delta\varphi_q\right| \simeq \frac{\sigma}{\varrho}, \tag{28}$$

and, respectively, its time derivative is of order

$$\left|\delta\dot{\varphi}_q\right| \simeq \frac{\sigma}{\varrho^2}, \tag{29}$$

where σ is the numerical coefficient of order unity (we set $\hbar = 1$). The quantum fluctuations grow very fast as $\varrho \to 0$ and, hence, Coleman's boundary condition $\dot{\varphi}_{\varrho=0} = 0$, formulated in the deep ultraviolet limit, must be re-analyzed. The potential $V(\varphi)$ can be arbitrarily shifted along the φ –axis. Therefore, to estimate the magnitude of quantum fluctuations it is more appropriate to consider either the typical change of the classical field $\Delta\varphi \simeq \dot{\varphi} \times \varrho$ in scales ϱ or it's time derivative. In both cases we obtain (up to a numerical coefficient) the same result and to be concrete we will use (29) to determine when quantum fluctuations become relevant. Equating the derivative of the classical solution (19) to the amplitude of quantum fluctuations (29) we find the following ultraviolet cutoff scale

$$\varrho_{uv} = \left(\frac{4\,\sigma}{\lambda_-\,\varphi_0^3}\right)^{1/3} = \left(\frac{\sigma^2}{32}\right)^{1/6} \lambda_-^{1/6}\,(1-\beta)^{1/2}\varrho_0, \tag{30}$$

where β is defined in (22) and ϱ_0 is given in (21). The classical solution with the Coleman boundary condition (7) is obviously valid only for $\varrho > \varrho_{uv}$ and it is completely spoiled by quantum fluctuations on scales smaller than ϱ_{uv}. Let us notice that for the very steep potentials with $\lambda_- \gg 1$ the ultraviolet cutoff scale exceeds the size of the bubble and therefore the instantons which lead to the large decay rate cannot be trusted anymore.

The ultraviolet cutoff ϱ_{uv} resulted from the fact that the quantum fluctuations are characterized by the field derivative of order $1/\varrho^2$ for all values of ϱ while the contribution of the classical solution at small ϱ decreases as ϱ. Thus, for small enough ϱ the quantum fluctuations dominate. For the large values of ϱ the quantum fluctuations continue to exhibit a $1/\varrho^2$ decay while the classical solution decreases as $1/\varrho^3$. Thus, the quantum fluctuations take over both in the deep ultra-violet and the large infrared scales carving a range of values of ϱ for which the classical solutions can be valid.

Next we estimate the infrared cutoff scale above which (in length) quantum fluctuations dominate. To find this scale we have to equate the derivative of solution (16), valid for $\varrho > \varrho_0$, to (29). As a result one gets

$$\varrho_{ir} \simeq \frac{2}{\sigma}\,\frac{\delta}{1+\delta}\,\varphi_0\,\varrho_0^2 \simeq \frac{4\sqrt{2}}{\sigma}\,\lambda_-^{-1/2}(1-\beta)^{1/2}\varrho_0. \tag{31}$$

To simplify the result we have assumed that $\varrho_{ir} \gg \varrho_0$. As one can check a posteriori this assumption is really valid when both λ_+ and λ_- are much smaller than unity.

For $\lambda_- > 1$ we have $\varrho_{ir} < \varrho_0$ and moreover $\varrho_{uv} > \varrho_{ir}$. Thus, the range

$$\varrho_{uv} < \varrho < \varrho_{ir} \tag{32}$$

where the classical instanton solution is supposed to be valid completely disappears. This tells us that for $\lambda_- > 1$ the classical instanton with the Coleman boundary condition can not be applied.

Concluding this section we would like to stress that both the *ultraviolet and infrared cutoff scales are entirely determined by the parameters characterizing the corresponding classical solution.*

5. New Instantons

The existence of the cutoff scales allows us to obtain a new class of $O(4)$ instantons, which otherwise would be singular and having an infinite action would not contribute to the decay rate. Let us first consider the new solutions emerging thanks to the existence of the ultraviolet cutoff scale. Later on we will estimate how the infrared cutoff corrections influence these new solutions. Thus, we will assume that for $\varrho > \varrho_0$ solution (16)–(17) is still valid, but abandon the boundary condition $\dot{\varphi}_{\varrho=0} = 0$. Then the most general solution of equation (18), which vanishes at $\varrho = \varrho_0$ and is valid for $\varphi < 0$, is

$$\varphi(\varrho) = \frac{\lambda_- \varphi_0^3}{8}\left(\varrho^2 - \varrho_0^2\right)\left(1 - \frac{\varrho_1^2}{\varrho^2}\right), \tag{33}$$

where the constant of integration $\varrho_1 < \varrho_0$ parametrizes our new instantons. These instantons are singular at $\varrho \to 0$. However, as we have shown above the solution (33) can only be trusted for $\varrho > \varrho_{uv}$. This allows us to avoid a singularity in the classical solution. When $\varrho_1 \neq 0$ there is a bounce at

$$\varrho_b = \sqrt{\varrho_0\, \varrho_1}, \tag{34}$$

where $\dot{\varphi}(\varrho_b) = 0$, and after that the field $\varphi(\varrho)$ must go back and vanish at ϱ_1, which is smaller than ϱ_b. However, before this happens the quantum fluctuations become relevant at $\varrho_{uv} > \varrho_b$ and as we will show the classical bubble with a quantum core emerges in Minkowski space. To calculate ϱ_{uv} we equate the derivative of (33) to (29) and obtain the following equation

$$1 - \frac{\varrho_b^4}{\varrho_{uv}^4} = \frac{4\sigma}{\lambda_- \varphi_0^3}\frac{1}{\varrho_{uv}^3}, \tag{35}$$

which can be solved exactly for ϱ_{uv}. Because we will not need the explicit solution for ϱ_{uv} we skip it here, but instead let us rewrite equation (35) in more convenient form as

$$\frac{\varrho_{uv}^2 - \varrho_b^2}{\varrho_{uv}^2} = \nu\left(\frac{\bar{\varrho}}{\varrho_{uv}}\right)^3, \tag{36}$$

where

$$\bar{\varrho}^2 \equiv \frac{8}{\varphi_0^2\, \lambda_-\, (1-\beta)} \tag{37}$$

and

$$\nu \equiv \frac{\sigma}{4\sqrt{2}\,\kappa\left(\varrho_{uv}\right)}\, \lambda_-^{1/2}\, (1-\beta)^{3/2}. \tag{38}$$

Here, $\kappa(\varrho_{uv}) = 1 + \varrho_b^2/\varrho_{uv}^2$ and since $\varrho_b^2/\varrho_{uv}^2 < 1$ it implies that $1 < \kappa(\varrho_{uv}) < 2$. The scale $\bar{\varrho}$ entering in (36) is equal to the bubble size ϱ_0 (see (21)) only for $\varrho_1 = 0$, corresponding to the Coleman instanton. To determine ϱ_0 in general case we require that $\dot{\varphi}(\varrho)$ is continuous at ϱ_0 and equating the derivatives of (16) and (33) at this point we obtain the following equation

$$1 + \frac{1}{\delta} = \frac{\lambda_- \varphi_0^2}{8}\left(\varrho_0^2 - \varrho_1^2\right), \tag{39}$$

where δ is given in (17). Solving this equation for ϱ_0^2 one gets

$$\varrho_0^2 = \frac{1}{2}\left(1 + \sqrt{1 + 4\beta\frac{\varrho_1^2}{\bar{\varrho}^2}}\right)\bar{\varrho}^2 + \varrho_1^2. \tag{40}$$

Thus we have found the new instantons with finite action, which are parametrized by ϱ_1. The main contribution to the decay rate comes from those instantons which have the minimal action while describing the required transition. Lets us stress that ϱ_1^2 must be positive. Had ϱ_1 been imaginary the solution of equation (33) would be valid all the way to ϱ equal zero. In that case the mode $\varphi \propto 1/\varrho^2$ would be increasing faster than the quantum fluctuations as $\varrho \to 0$ and we would end up at a singularity, where $\varphi \to -\infty$. For positive ϱ_1^2 the classical field φ bounces at ϱ_b and evolves towards the positive values. However, as we have noticed above, before the bounce is reached the classical solution fails at $\varrho_{uv} > \varrho_b$ and the bubble emerges from under the barrier materializing in Minkowski space.

To simplify the formulae it is convenient to parametrize the instantons by the dimensionless parameter $\chi(\varrho_1)$, related to ϱ_1^2 as

$$\varrho_1^2 = \chi(1 + \beta\chi)\bar{\varrho}^2, \tag{41}$$

where $\bar{\varrho}^2$ is defined in (37). Then, as it follows from (40)

$$\varrho_0^2 = (1+\chi)(1+\beta\chi)\bar{\varrho}^2 \tag{42}$$

and

$$\varrho_b^2 = \varrho_0\,\varrho_1 = (1+\beta\chi)\sqrt{\chi(1+\chi)}\,\bar{\varrho}^2. \tag{43}$$

The expression (33) for $\varphi(\varrho)$ can be rewritten in the following more convenient form:

$$\varphi(\varrho) = -\frac{\varphi_0}{1-\beta}\left(\frac{\left(\varrho_0 - \varrho_1\right)^2}{\bar{\varrho}^2} - \frac{\left(\varrho^2 - \varrho_b^2\right)^2}{\bar{\varrho}^2\varrho^2}\right) \tag{44}$$

and using the formulae above as well as equation (36) we obtain

$$\varphi_{uv} \equiv \varphi\left(\varrho_{uv}\right) = -\frac{\varphi_0}{1-\beta}\left(\frac{1+\beta\chi}{\left(\sqrt{1+\chi} + \sqrt{\chi}\right)^2} - \nu^2\left(\frac{\bar{\varrho}}{\varrho_{uv}}\right)^4\right). \tag{45}$$

The value of the potential at φ_{uv} is

$$V_{uv}(\chi) = V\left(\varphi_{uv}\right) = -\frac{4V_{bar}}{\beta}\left(\frac{1+\beta\chi}{\left(\sqrt{1+\chi}+\sqrt{\chi}\right)^2} - \frac{\beta}{4} - \nu^2\left(\frac{\bar{\varrho}}{\varrho_{uv}}\right)^4\right). \tag{46}$$

Assuming that $\lambda_- \ll 1$ we find that when χ varies from zero to infinity V_{uv} changes within the range

$$-\frac{4V_{bar}}{\beta}\left(1-\frac{\beta}{4}-\nu^{2/3}\right) < V_{uv}(\chi) < 0, \tag{47}$$

vanishing at $\chi \to \infty$. The last term in the brackets is due to the quantum corrections and it is much smaller than unity for $\lambda_- \ll 1$ (the case of $\lambda_- > 1$ will be considered separately).

Thus, we conclude that as a result of quantum regularization there appears a whole class of new nonsingular instantons which all contribute to the decay of the false vacuum. Depending on χ, which parametrizes these instantons, the value of the potential in the central part of the bubble dominated by quantum fluctuations, after subtracting the energy of zero point fluctuations, is equal to V_{uv} and takes its value within the interval (47). If there exists a second true vacuum with $V_{\min}$ in the range (47) (see curve (b) in Figure 2), we expect that the dominant contribution to the false vacuum decay is given by the instanton with χ determined by equation $V_{uv}(\chi) \simeq V_{\min}$.

To verify that the new instanton really emerges from under the potential barrier in Minkowski space at $\tau = 0$ we have to check that

$$\mathcal{V}(\tau=0) = 4\pi\int_{\varrho_{uv}}^{+\infty} d\varrho\,\varrho^2\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) + \frac{4\pi}{3}\varrho_{uv}^3 V_{uv} \tag{48}$$

vanishes up to a possible quantum correction not exceeding the contribution of a single quantum. The second term in (48) accounts for the shifted energy inside the bubble quantum core, which remains after normalizing the energy of quantum fluctuations to zero in the false vacuum state. Substituting solutions (16) and (33) in (48) we obtain

$$\begin{aligned}
&4\pi\int_{\varrho_{uv}}^{+\infty} d\varrho\,\varrho^2\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)\\
&= \frac{\pi\lambda_-^2\varphi_0^6}{24}\left[\frac{\left(\varrho_{uv}^4-\varrho_b^4\right)^2}{\varrho_{uv}^3} - 4\varrho_{uv}\left(\left(\varrho_0^2-\varrho_{uv}^2\right)\left(\varrho_1^2-\varrho_{uv}^2\right)\right.\right.\\
&\left.\left.+\frac{2\lambda_+}{\lambda_-^2\varphi_0^2}\varrho_{uv}^2\right)\right]\\
&= \frac{4\pi}{3}\varrho_{uv}^3\left(\frac{1}{2}\left(\dot{\varphi}(\varrho_{uv})\right)^2 - V\left(\varphi(\varrho_{uv})\right)\right),
\end{aligned} \tag{49}$$

and hence

$$\mathcal{V}(\tau=0) = \frac{2\pi}{3}\varrho_{uv}^3\left(\dot{\varphi}(\varrho_{uv})\right)^2 = \frac{2\pi\sigma^2}{3\varrho_{uv}}, \tag{50}$$

which corresponds to one quantum of energy in scales ϱ_{uv}. Thus, the energy balance is satisfied within an accuracy dictated by the time-energy uncertainty relation and the bubble emerges from under the barrier.

To determine the decay rate we calculate the action

$$S_I = 2\pi^2\int_{\varrho_{uv}}^{+\infty} d\varrho\,\varrho^3\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) + \frac{\pi^2}{2}\varrho_{uv}^4 V_{uv} \tag{51}$$

where the last term accounts for the contribution of the bubble quantum core. The result is

$$\begin{aligned}
S_I = \frac{\pi\varphi_0^2}{96}&\left[\lambda_-^2\varphi_0^4\left(\varrho_{uv}^6 + \frac{3\varrho_b^8}{\varrho_{uv}^2} - \varrho_0^6 - \frac{3\varrho_b^8}{\varrho_0^2}\right)\right.\\
&+\frac{32\left(\lambda_-\varphi_0^2\left(\varrho_0^2-\varrho_1^2\right)-8\right)}{\lambda_+\varphi_0^2}\\
&\left.+\left(16\,\lambda_-\varphi_0^2\left(\left(1+\frac{3\lambda_+}{4\lambda_-}\right)\varrho_0^2-\varrho_1^2\right)\right)\varrho_0^2 + 32\,\varrho_0^2\right].
\end{aligned} \tag{52}$$

Substituting here the expression for ϱ_0, ϱ_1 and ϱ_b from (41)–(43) and using equation (36) for ϱ_{uv} this action can be represented in the following form:

$$\begin{aligned}
S_I = \frac{8\pi^2}{3\lambda_-(1-\beta)^3}&\left[(2-\beta)(2-2\beta+\beta^2)\right.\\
&+\frac{4\chi^{3/2}(4+3\chi)(1+\beta\chi)^3}{2(1+\chi)^{3/2}+\chi^{1/2}(3+2\chi)}\\
&\left.-\beta\chi^2\left(1+2(1+\beta\chi)+3(1+\beta\chi)^2\right)\right]\\
&+\frac{\pi^2\sigma^2}{6}\left(1+2\left(\frac{\varrho_b^2}{\varrho_{uv}^2+\varrho_b^2}\right)^2\right).
\end{aligned} \tag{53}$$

The last term here is always of order unity and can be neglected.

Remarks on infrared cutoff. Considering instantons we have ignored the infrared cutoff. Let us estimate how the results above are changed if we take it into account. For $\chi \neq 0$ the expression (31) for ϱ_{ir} is modified as

$$\varrho_{ir} \simeq \frac{4\sqrt{2}}{\sigma}\lambda_-^{-1/2}(1-\beta)^{1/2}\left(\frac{1+\chi}{1+\beta\chi}\right)^{1/2}\varrho_0. \tag{54}$$

If both $\lambda_+, \lambda_- \ll 1$ the bubble size ϱ_0 is always much smaller than ϱ_{ir} and the infrared effects do not influence much the instantons for any χ. However, if $\lambda_- \gg 1$, it follows from (54) that only if

$$\chi \gg \chi_{\min} = 4\,\nu^2, \tag{55}$$

where ν is defined in (38), we have $\varrho_{ir} \gg \varrho_0$, otherwise ϱ_{ir} can becomes even smaller than ϱ_{uv} and, hence, the classical solutions with $\chi < \chi_{\min}$ do not make sense. Thus, for the steep potentials with $\lambda_- \gg 1$ the value of χ must always exceed $\chi_{\min} \approx \lambda_-$. If not, even with the best of intentions, the spirit of the semi-classical approximation can not be retained and the quantum effects take over.

Calculating the corrections to the potential energy (50) and action (51) due to the infrared cutoff, we find that

$$\begin{aligned}
\Delta\mathcal{V}_{ir} &= -4\pi\int_{\varrho_{ir}}^{+\infty} d\varrho\,\varrho^2\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)\\
&= \frac{2\pi\sigma^2}{3}\left(1+\frac{\sigma^2\lambda_+}{60}\right)\frac{1}{\varrho_{ir}}
\end{aligned} \tag{56}$$

and

$$\Delta S_I = -\frac{\pi^2 \sigma^2}{32}\left(1 + \frac{\sigma^2 \lambda_+}{4}\right), \tag{57}$$

respectively. If we would decide to normalize $V(\varphi(\varrho_{ir}))$ to zero we would have to subtract from the action

$$\Delta S = -\frac{\pi^2 \sigma^4 \lambda_+}{128}. \tag{58}$$

Thus, as it follows from (56)–(58) all infrared corrections are at the level of one quantum with the frequency corresponding to the infrared cutoff scale. Hence, except the important bound (55) which has to be respected for $\lambda_- \gg 1$, the quantum infrared corrections can be ignored.

6. Implications

We will now apply the results obtained above in several limiting cases. We consider separately the instantons with $\chi \ll 1$ and $\chi \gg 1$. The action (53) simplifies to

$$S_I(\chi) = \frac{8\pi^2}{3\lambda_-(1-\beta)^3}\left[(2-\beta)(2-2\beta+\beta^2) + 8\chi^{3/2} + O(\chi^2)\right] \tag{59}$$

for $\chi \ll 1$ and up to corrections of order $\chi^{3/2}$ coincides with the Coleman action, while for $\chi \gg 1$ it becomes

$$S_I = \frac{8\pi^2 \chi}{\lambda_-(1-\beta)^3}\left[\frac{1}{3}\left(1-\frac{\beta}{2}\right)(\beta\chi)^2 + \left(1-\frac{\beta}{4}\right)^2 \beta\chi + \left(1-\frac{\beta}{4}\right)\left(1-\frac{\beta}{4}+\frac{\beta^2}{8}\right) + O\left(\frac{1}{\chi}\right)\right]. \tag{60}$$

Let us recall that in order to ignore the quantum loop corrections λ_+ must always be smaller than unity, while λ_- can be large and therefore we investigate the potentials with $\lambda_- \ll 1$ and $\lambda_- \gg 1$ separately.

A) For $\lambda_- \ll 1$ the parameter ν defined in (38) is much smaller than unity and thus from (46) one obtains

$$V_{uv}(\chi) = -\frac{4V_{bar}}{\beta}\left(1 - \frac{\beta}{4} - 2\sqrt{\chi} - \nu^{2/3} + O(\chi)\right) \tag{61}$$

for $\chi \ll 1$. These instantons with the action (59) give the main contribution to the decay rate for unbounded potential and for the potentials with the second true minimum of depth $V_{\min} < V_{uv}(\chi = 0)$. They are different from the Coleman instanton ($\chi = 0$) only by higher order corrections. As it follows from (36) the size of quantum core

$$\varrho_{uv} \simeq \left(\frac{\sigma^2}{32}\right)^{1/6} \lambda_-^{1/6}(1-\beta)^{1/2}\varrho_0 \tag{62}$$

is much smaller than the bubble size

$$\varrho_0 = \left(1 + \frac{1+\beta}{2}\chi + O(\chi^2)\right)\bar{\varrho}. \tag{63}$$

To calculate the contribution of instantons with the value of the potential in the quantum core

$$V_{uv}(\chi_\varepsilon) = \varepsilon V_{uv}(\chi = 0) \simeq -\frac{4\varepsilon V_{bar}}{\beta}\left(1 - \frac{\beta}{4}\right), \tag{64}$$

where $\varepsilon \ll 1$, to the decay rate we have to consider solutions with $\chi \gg 1$. In this case the expression (46) simplifies to

$$V_{uv}(\chi) = -\frac{V_{bar}}{\beta}\left(\left(1-\frac{\beta}{2}\right)\frac{1}{\chi} + O\left(\frac{1}{\chi^2}\right)\right), \tag{65}$$

and as it follows from (64)

$$\chi_\varepsilon \simeq \frac{1}{2}\left(\frac{2-\beta}{4-\beta}\right)\frac{1}{\varepsilon} \gg 1. \tag{66}$$

The expressions for ϱ_0^2 becomes

$$\varrho_0^2(\chi_\varepsilon) \simeq \frac{4}{(1-\beta)}\left(\frac{2-\beta}{4-\beta}\right)^2 \frac{1}{\varepsilon\lambda_-\varphi_0^2}\left(\left(\frac{4-\beta}{2-\beta}\right) + \frac{1}{2}\frac{\beta}{\varepsilon}\right), \tag{67}$$

and the action (60) is equal to

$$S_\varepsilon = \frac{\pi^2(2-\beta)}{\varepsilon\lambda_-(1-\beta)^3}\left[\frac{1}{6}\left(\frac{2-\beta}{4-\beta}\right)^3\frac{\beta^2}{\varepsilon^2} + \frac{(2-\beta)}{8}\frac{\beta}{\varepsilon} + \left(1 - \frac{\beta}{4} + \frac{\beta^2}{8}\right) + O(\varepsilon)\right]. \tag{68}$$

The contribution of these χ_ε-instantons to the decay rate is given by

$$\Gamma \sim \varrho_0^{-4}(\chi_\varepsilon)\exp(-S_\varepsilon). \tag{69}$$

For the unbounded potential the instantons with $\chi \ll 1$ give the main contribution to the overall decay rate. However, even in this case the whole spectrum of instantons is present when we consider the false vacuum decay. The formula (69) can also be applied to estimate the leading contribution to the decay rate for a broad class of potentials with two minima in the case when the potential is well approximated by the linear potential nearly up to the second true minimum of depth

$$V_{\min} \simeq -\frac{4\varepsilon V_{bar}}{\beta}\left(1 - \frac{\beta}{4} - \nu^{2/3}\right) \tag{70}$$

(see, for example, curve (b) in Figure 2).

The expression (17) for the parameter δ, characterizing the thickness of the bubble wall for $\chi \neq 0$, is modified as

$$\delta = \frac{1-\beta}{\beta(1+\chi)}, \tag{71}$$

and hence the bubble has a thin wall $\delta \ll 1$ for all χ if $\lambda_- \ll \lambda_+$ $(1-\beta \ll 1)$. In the case $\lambda_+ \ll \lambda_- < 1$ only bubbles with $\chi_\varepsilon \sim \varepsilon^{-1} \gg \lambda_-/\lambda_+$ have a thin wall.

When we have two nearly degenerate minima, that is $\varepsilon \to 0$ $(\varepsilon \ll \beta)$, the probability of the vacuum decay, for example, for $\beta \ll 1$ vanishes as

$$\Gamma \sim 4\lambda_+^2 \left(\varphi_0 \frac{\varepsilon}{\beta}\right)^4 \exp\left[-\frac{\pi^2}{24\lambda_+}\left(\frac{\beta}{\varepsilon}\right)^3\right] \tag{72}$$

in complete agreement with our expectations.

B) $\lambda_- \gg 1$ (*zero size instanton problem*). Finally let us consider the most interesting case of a very steep unbounded potential (see Figure 3 where $\lambda_- \to \infty$). Since λ_+ is always smaller than unity $\beta \ll 1$ and the parameter ν^2 in (38) is

$$\nu^2 \simeq \frac{\sigma^2}{128}\lambda_- \gg 1. \tag{73}$$

As one can find by inspecting (45) and (46), in this case the value of χ must be larger than unity because otherwise the quantum fluctuations dominate over the classical instanton solution everywhere. Therefore, the Coleman instanton $(\chi = 0)$ is never trustable. The solution of equation (36) is

$$\varrho_{uv} = \varrho_b\left(1 + \frac{1}{2}\frac{\nu}{((1+\beta\chi)\chi)^{3/2}} + O\left(\frac{\nu^2}{\chi^3}\right)\right) \tag{74}$$

and to the leading order the equations (45) and (46) simplify to

$$\varphi_{uv}(\chi) \simeq -\varphi_0\left(\frac{1+\beta\chi}{4\chi} - \frac{\nu^2}{(1+\beta\chi)^2\chi^2}\right) \tag{75}$$

and

$$V_{uv}(\chi) \simeq -\frac{4V_{bar}}{\beta}\left(\frac{1}{4\chi} - \frac{\nu^2}{(1+\beta\chi)^2\chi^2}\right). \tag{76}$$

It follows from here that only if

$$\chi > 4\nu^2 \tag{77}$$

both φ_{uv} and V_{uv} are negative and hence the tunneling becomes possible. This lower bound on χ is in complete agreement with the bound (55) obtained by inspecting the relevance of the infrared cutoff scale. Because of this bound the minimal value of the potential in the center of the bubble must be always larger than

$$V_c \simeq -\frac{32\varphi_0^4}{\sigma^2}. \tag{78}$$

Let us consider the χ_ε instanton for which

$$V_{uv}(\chi_\varepsilon) = \varepsilon V_c, \tag{79}$$

where $\varepsilon < 1$. We obtain the corresponding χ_ε substituting expansion (76) into (79), where we keep only the first term,

$$\chi_\varepsilon \simeq \frac{\sigma^2\lambda_-}{128\varepsilon}. \tag{80}$$

The size of the bubble is given by

$$\varrho_0^2\varphi_0^2 \simeq \frac{\sigma^2}{16\varepsilon}\left(1 + \frac{\sigma^2\lambda_+}{128\varepsilon} + O\left(\frac{\varepsilon}{\lambda_-}\right)\right). \tag{81}$$

and this size is always larger than some minimal size, that is,

$$\varrho_0 \geq \frac{\sigma}{4\varphi_0}, \tag{82}$$

irrespectively of the value of $\lambda_- > 1$. This resolves the problem of small size instantons for the steep potentials. To calculate the action we have to substitute (80) in (60). As a result one gets

$$S(\chi_\varepsilon) \simeq \frac{\pi^2\sigma^2}{16\varepsilon}\left[1 + \frac{\sigma^2\lambda_+}{128\varepsilon} + \frac{1}{3}\left(\frac{\sigma^2\lambda_+}{128\varepsilon}\right)^2 + O\left(\frac{\varepsilon}{\lambda_-}\right)\right]. \tag{83}$$

Notice that both the bubble size or the action do not depend on λ_- in the leading order and therefore we can take a limit $\lambda_- \to \infty$ (see Figure 3), when all λ_- -dependent corrections proportional to ε/λ_- vanish. Although all instantons contribute to the vacuum decay rate, that major contribution to this rate for the unbounded potential and the potentials with the second very deep minimum $V_{\min} < V_c$ (see curve (a) in Figure 3) give instantons with the minimal possible size and action, corresponding to $\varepsilon \simeq 1$, so that

$$\Gamma \sim \varphi_0^4 \exp\left[-\frac{\pi^2\sigma^2}{16}\right]. \tag{84}$$

One has to stress that in this case we are working on the border of semiclassical approximation and the emerging bubbles contain only one quantum. Therefore equation (84) gives only a very rough estimate for the probability of decay. To determine the leading contribution to the false vacuum decay rate in case when the second minimum of depth $V_{\min}$ is above V_c (see curve (b) in Figure 3) one has to use the formulae (81) and (83) with ε determined by equation $V_{\min} \simeq \varepsilon V_c$.

7. Conclusions

In this paper we have analyzed the role of quantum fluctuations for the classical $O(4)$ solutions which describe the false vacuum decay. It was shown that the quantum fluctuations induce the ultraviolet and infrared cutoff scales, beyond which the classical instanton solutions cannot be trusted anymore because the level of quantum fluctuations exceed the contribution of the classical field. The cutoff scales are entirely determined by the parameters characterizing the corresponding classical solutions. This allows one to abandon the boundary condition imposed by Coleman on the derivative of the field at $\rho = 0$, which is outside the range of validity of the approximation. The regularized solutions are indeed not singular. As a result there emerges the whole spectrum of new instantons, which all contribute to the false vacuum decay. The largest contribution comes from the instantons with the minimal possible action. In many cases these instantons (up to small corrections) lead to the same result as instanton with the Coleman boundary conditions. However, in several important cases the Coleman approach fails. In particular, we have shown that for the very steep unbounded from below potential, shown in Figure 3, the instanton with the Coleman boundary conditions would lead to nearly instantaneous vacuum instability via formation of infinitely small bubbles. To the contrary, as the size of the new instantons always exceeds some minimal scale determined by the parameters of the original potential, the decay probability obtained by these new instantons remains finite even for case where the unbounded potential is very steep. This removes the difficulties that were associated with the application of small instantons. We have also checked that the results in the method we suggest agrees with the standard method when they overlap.

Acknowledgements

The work of V. M. was supported under Germany's Excellence Strategy—EXC-2111—Grant No. 39081486.

The work of E. R. is partially supported by the Israeli Science Foundation Center of Excellence. E. R. would also like to thank NHETC at Rutgers Physics Department, the IHES and the CCPP at NYU for hospitality.

A. S. would like express a gratitude to the Racah Institute of Physics of the Hebrew University of Jerusalem and Ludwig Maxmillian University of Munich for the hospitality during his visits. The work of A. S. was partially supported by RFBR grant No. 20-02-00411.

Conflict of Interest

The authors have declared no conflict of interest.

Keywords

false vacuum decay, instanton, quantum fluctuations

Received: October 11, 2020
Published online:

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